Proof of Helmholtz's Vortex Theorems

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December 26, 2024

Abstract

Derivations or proofs of Helmholtz's vortex theorems for fluid flows in textbooks are often rudimentary oder purely intuitive and thus not altogether convincing. In this Tutorial the vortex theorems are exactly formulated und mathematically completely proven.

1 Formulation of the vortex theorems

1.1 Prerequisites

We consider the flow of an inviscid barotropic fluid. External force fields, if present, are assumed to be conservative. If ρ is the mass density, $\vec{v}$ is the flow velocity, p is the pressure, $\vec{f}_{\rm ext}$ is the spatial density of the external forces and

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \vec{v}\,\nabla \tag{1}$$

denotes the Lagrangian (or material or mobile) time derivative, then the flow obeys the Euler equation

$$\rho\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = -\nabla p + \vec{f}_{\rm ext} \tag{2}$$

with the constraints

$$\nabla\times\frac{\nabla p}{\rho} = \vec{0}\,, \quad \nabla\times\frac{\vec{f}_{\rm ext}}{\rho} = \vec{0} \tag{3}$$

and the equation of mass conservation (continuity equation)

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho\,\nabla\vec{v}\,. \tag{4}$$

The first of the constraints in Eq. (3) defines the property of barotropicity. It is obviously satisfied in the special case of a spatially homogeneous mass density, $\rho \equiv \rho_0$, since $1/\rho$ can then be pulled out of the differentiation by the operator $\nabla\times$.

This includes the much studied case of an incompressible flow, characterized by the condition $\nabla\vec{v} = 0$, with spatially and because of Eq. (4) then also temporally constant mass density.

The first of the constraints in Eq. (3) is also satisfied in the more general case that ρ varies spatially but depends solely on pressure, $\rho = \rho(p)$, since then

$$\nabla \times \frac{\nabla p}{\rho} = \frac{1}{\rho}\nabla \times \nabla p + \nabla\frac{1}{\rho} \times \nabla p = \nabla\frac{1}{\rho} \times \nabla p = -\frac{1}{\rho^2}\frac{\partial\rho}{\partial p}\nabla p \times \nabla p = \vec{0}\,. \qquad (5)$$

An example of $\rho = \rho(p)$ is the polytropic equation of state $p = K\rho^\gamma$, where K and γ are constants, with the inversion $\rho = (p/K)^{-\gamma}$. This includes the special case of an adiabatic ideal gas, where $\gamma = c_\mathrm{p}/c_\mathrm{v}$ is the ratio of the specific heats at constant pressure and constant volume, respectively.

An important example of external forces is gravity. Let $\vec{g}$ be the gravitational acceleration (the strength of the gravitational field, defined as force per unit volume). In the absence of further external forces, we have $\vec{f}_\mathrm{ext}/\rho = \vec{g}$ and the second constraint in Eq. (3) is satisfied since the gravitational field of an arbitrary mass distributions is conservative — the associated gravitational potential U (so that $\vec{g} = -\nabla U$) can be determined from the Poisson equation $\Delta U = 4\pi G\rho$, where G is the gravitational constant. In models where the acceleration due to gravity near the surface of the Earth is assumed spatially constant, the fulfillment of the condition $\nabla \times (\vec{f}_\mathrm{ext}/\rho) = \nabla \times \vec{g} = \vec{0}$ is obvious.

The Lagrangian time derivative, defined by Eq. (1), is the total time derivative along the path of a fluid particle moving with the flow. In the Lagrangian description, the position $\vec{x}$ of a fluid particle is considered as a function of its initial position $\vec{x}_0$ and time t, that is, $\vec{x} = \vec{x}(\vec{x}_0, t)$. The index 0 indicates initial values (at $t = 0$). We assume the maps of the initial positions on the actual positions to be diffeomorphisms, that is, each of these maps (at a fixed time t) is assumed to be uniquely invertible (bijective) and, as well as its inverse, continuously differentiable (with respect to $\vec{x}_0$ and $\vec{x}$, respectively). In addition, $\vec{x}(\vec{x}_0, t)$ is assumed to be continuously differentiable with respect to the parameter t.

The assumed continuity of the motion of the fluid particles has the following implications:
(i) Curves and surfaces formed by fluid particles remain curves and surfaces.
(ii) Connected fluid volumes remain connected.
(iii) The boundary of a fluid volume remains boundary; interior fluid particles stay in the interior.

The vorticity is defined by

$$\vec{\omega} = \nabla \times \vec{v}\,. \qquad (6)$$

Applying the operator $\nabla\times$ to Eq. (2), taking into account the conditions in Eq. (3) and employing the vector identity

$$\frac{1}{2}\nabla\vec{v}^2 = \vec{v} \times (\nabla \times \vec{v}) + (\vec{v}\nabla)\vec{v}\,, \qquad (7)$$

we get the vorticity equation

$$\frac{\partial\vec{\omega}}{\partial t} = \nabla \times (\vec{v} \times \vec{\omega})\,. \qquad (8)$$

Throughout this tutorial, vectors are considered as column vectors or matrices with only one column. Their transposes, indicated by a superscripted t, are then row vectors or matrices with only one row. Using index notation, repeated use is made of the Kronecker tensor

$$\delta_{ij} = \begin{cases} 1\,, & \text{if } i = j \\ 0\,, & \text{if } i \neq j \end{cases} \tag{9}$$

and the Levi-Civita tensor

$$\epsilon_{ijk} = \begin{cases} 1\,, & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3) \\ -1\,, & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3) \\ 0\,, & \text{if any two of the indices } i,j,k \text{ are equal} \end{cases} \tag{10}$$

and, according to the Einstein summation convention, indices appearing twice are implicitly summed over.

1.2 First Theorem

The field lines of $\vec{\omega}$ are called vortex lines. Each vortex line defines a material curve (a fluid thread) formed by the fluid particles that are located along this vortex line.

Helmholtz's first vortex theorem states: The fluid particles that are located along a vortex line (that form a material curve or fluid thread coinciding with a vortex line) at some instant of time retain this property. The vortex lines are thus said to "move with the fluid" or to be "materialized", "conserved", "frozen into the fluid".

This conservation of the vortex lines immediately carries over to the flux tubes of $\vec{\omega}$, called vortex tubes, which can be defined as bundles of vortex lines crossing a (finite) surface to which $\vec{\omega}$ is nowhere tangential; that $\vec{\omega}$ remains non-tangential when the surface moves with the fluid is ensured by Helmholtz's second vortex theorem (see Section 1.3).

1.3 Second Theorem

Let C be a closed curve formed by fluid particles (a closed material curve). The line integral

$$\Gamma = \oint_C \vec{v}\,\mathrm{d}\vec{x} \tag{11}$$

is called the circulation around C. According to the theorem of Stokes, we also have

$$\Gamma = \int_S \vec{\omega}\,\mathrm{d}\vec{S}\,, \tag{12}$$

where S is any surface spanned by C (any surface with boundary curve C).

Helmholtz's second vortex theorem states that

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = 0\,. \tag{13}$$

The circulation around a closed curve moving with the fluid is conserved. According to Eq. (12), this is equivalent to stating that the flux of $\vec{\omega}$ through a surface moving with the fluid is conserved.

This second Helmholtz's vortex theorem is also known as Kelvin's theorem.

2 Velocity and Deformation Gradient Matrices

Let V be the velocity gradient matrix,

$$V_{ij} = \frac{\partial v_i}{\partial x_j}\,, \tag{14}$$

and J be the deformation gradient matrix,

$$J_{ij} = \frac{\partial x_i}{\partial x_{0j}}\,. \tag{15}$$

J is the Jacobian matrix of the map $\vec{x}_0 \to \vec{x}$. Since this map is invertible, the determinant of J cannot vanish, thus cannot change its sign. Furthermore

$$J_{ij} = \delta_{ij}, \quad \det(J) = 1 \quad \text{at } t = 0\,, \tag{16}$$

so that

$$0 < \det(J) < \infty\,. \tag{17}$$

For the Lagrangian time derivative of J we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial x_i}{\partial x_{0j}} = \frac{\partial}{\partial x_{0j}}\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{\partial v_i}{\partial x_{0j}} = \frac{\partial v_i}{\partial x_p}\frac{\partial x_p}{\partial x_{0j}} = V_{ip}J_{pj}\,, \tag{18}$$

which can also be written as

$$\frac{\mathrm{d}J}{\mathrm{d}t} = VJ\,. \tag{19}$$

By using

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}(JJ^{-1}) = \frac{\mathrm{d}J}{\mathrm{d}t}J^{-1} + J\frac{\mathrm{d}J^{-1}}{\mathrm{d}t} = VJJ^{-1} + J\frac{\mathrm{d}J^{-1}}{\mathrm{d}t} = V + J\frac{\mathrm{d}J^{-1}}{\mathrm{d}t} \tag{20}$$

we furthermore find

$$\frac{\mathrm{d}J^{-1}}{\mathrm{d}t} = -J^{-1}V\,. \tag{21}$$

3 Infinitesimal Line, Surface and Volume Elements

3.1 Infinitesimal Line Element

The Lagrangian change of a line element $\mathrm{d}\vec{x}$ is found by forming the total differential of the function $\vec{x}(\vec{x}_0)$ at a fixed instant of time, namely,

$$\mathrm{d}x_i = \frac{\partial x_i}{\partial x_{0j}}\mathrm{d}x_{0j} \tag{22}$$

or, equivalently,

$$\mathrm{d}\vec{x} = J\mathrm{d}\vec{x}_0\,. \tag{23}$$

3.2 Infinitesimal Surface Element

Let the oriented surface element $\mathrm{d}\vec{\sigma}$ be spanned by the two line elements $\mathrm{d}\vec{x}^{(1)}$ and $\mathrm{d}\vec{x}^{(2)}$,

$$\mathrm{d}\vec{\sigma} = \mathrm{d}\vec{x}^{(1)} \times \mathrm{d}\vec{x}^{(2)} = J\mathrm{d}\vec{x}_0^{(1)} \times J\mathrm{d}\vec{x}_0^{(2)} \,. \tag{24}$$

In index notation this yields

$$\begin{aligned}
\mathrm{d}\sigma_i &= \varepsilon_{ijk} \left(J\mathrm{d}\vec{x}_0^{(1)}\right)_j \left(J\mathrm{d}\vec{x}_0^{(2)}\right)_k \\
&= \varepsilon_{ijk} J_{jn} \mathrm{d}x_{0n}^{(1)} J_{km} \mathrm{d}x_{0m}^{(2)} \\
&= \delta_{il} \varepsilon_{ljk} J_{jn} \mathrm{d}x_{0n}^{(1)} J_{km} \mathrm{d}x_{0m}^{(2)} \\
&= \left(J^{\mathrm{t}}\right)^{-1}_{ip} J^{\mathrm{t}}_{pl}\, \varepsilon_{ljk} J_{jn} \mathrm{d}x_{0n}^{(1)} J_{km} \mathrm{d}x_{0m}^{(2)} \\
&= \left(J^{\mathrm{t}}\right)^{-1}_{ip} \varepsilon_{ljk} J_{lp} J_{jn} J_{km} \mathrm{d}x_{0n}^{(1)} \mathrm{d}x_{0m}^{(2)} \\
&= \left(J^{\mathrm{t}}\right)^{-1}_{ip} \det(J) \varepsilon_{pnm} \mathrm{d}x_{0n}^{(1)} \mathrm{d}x_{0m}^{(2)} \,,
\end{aligned} \tag{25}$$

which, now again using vector notation, can be summarized in the form

$$\mathrm{d}\vec{\sigma} = \det(J) \left(J^{-1}\right)^{\mathrm{t}} \mathrm{d}\vec{\sigma}_0 \,. \tag{26}$$

3.3 Infinitesimal Volume Element

Let, finally, an infinitesimal volume element be spanned by the three line elements $\mathrm{d}\vec{x}^{(1)}$, $\mathrm{d}\vec{x}^{(2)}$ and $\mathrm{d}\vec{x}^{(3)}$, supposed to form a right-handed system. Their scalar triple product yields the content $\mathrm{d}\mathcal{V}$ of the volume element:

$$\begin{aligned}
\mathrm{d}\mathcal{V} &= \left(\mathrm{d}\vec{x}^{(1)} \times \mathrm{d}\vec{x}^{(2)}\right)^{\mathrm{t}} \mathrm{d}\vec{x}^{(3)} \\
&= \mathrm{d}\vec{\sigma}^{\mathrm{t}} \mathrm{d}\vec{x}^{(3)} \\
&= \det(J) \mathrm{d}\vec{\sigma}_0^{\mathrm{t}} J^{-1} J \mathrm{d}\vec{x}_0^{(3)} \\
&= \det(J)\, \mathrm{d}\mathcal{V}_0
\end{aligned} \tag{27}$$

For the mass density ρ, it follows

$$\rho = \frac{\rho_0}{\det(J)} \,. \tag{28}$$

4 Transport Equation for $\vec{\omega}/\rho$

From the vorticity equation for an inviscid and barotropic flow, Eq. (8), combined with the equation of mass conservation, Eq. (4), using the substitution of the Lagrangian time derivative according to Eq. (1), as well as the vector identity

$$\nabla \times (\vec{v} \times \vec{\omega}) = (\vec{\omega}\nabla)\vec{v} - (\vec{v}\nabla)\vec{\omega} + \vec{v}(\nabla\vec{\omega}) - \vec{\omega}(\nabla\vec{v}) \,, \tag{29}$$

where the third summand on the right-hand side vanishes because of $\nabla\vec{\omega} = \nabla(\nabla \times \vec{v}) = 0$, we get:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\vec{\omega}}{\rho} &= \frac{1}{\rho}\frac{\mathrm{d}\vec{\omega}}{\mathrm{d}t} - \frac{\vec{\omega}}{\rho^2}\frac{\mathrm{d}\rho}{\mathrm{d}t} \\ &= \frac{1}{\rho}\left(\frac{\partial\vec{\omega}}{\partial t} + (\vec{v}\nabla)\vec{\omega}\right) + \frac{\vec{\omega}}{\rho}\nabla\vec{v} \\ &= \frac{1}{\rho}\left(\nabla \times (\vec{v} \times \vec{\omega}) + (\vec{v}\nabla)\vec{\omega}\right) + \frac{\vec{\omega}}{\rho}\nabla\vec{v} \\ &= \left(\frac{\vec{\omega}}{\rho}\nabla\right)\vec{v} \end{aligned} \tag{30}$$

Equations (21) and (30) yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(J^{-1}\frac{\vec{\omega}}{\rho}\right) = -J^{-1}V\frac{\vec{\omega}}{\rho} + J^{-1}\left(\frac{\vec{\omega}}{\rho}\nabla\right)\vec{v}\,, \tag{31}$$

from which, taking into account

$$\left[\left(\frac{\vec{\omega}}{\rho}\nabla\right)\vec{v}\right]_i = \frac{\omega_j}{\rho}\nabla_j v_i = \frac{\omega_j}{\rho}V_{ij} = \left[V\frac{\vec{\omega}}{\rho}\right]_i\,, \tag{32}$$

the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(J^{-1}\frac{\vec{\omega}}{\rho}\right) = \vec{0} \tag{33}$$

is obtained, which can also be written in the form

$$J^{-1}\frac{\vec{\omega}}{\rho} = J^{-1}(t=0)\frac{\vec{\omega_0}}{\rho_0} \tag{34}$$

and, because of $J^{-1}(t=0)_{ij} = \delta_{ij}$, finally in the form

$$\frac{\vec{\omega}}{\rho} = J\frac{\vec{\omega_0}}{\rho_0}\,. \tag{35}$$

The vortex theorems can be derived from Eq. (35) with the aid of the relations in Sec. 3.

5 Derivation of the Two Vortex Theorems from the Transport Equation for $\vec{\omega}/\rho$

5.1 First Theorem: Conservation of the Vortex Lines

The conservation of the field lins of $\vec{\omega}$ immediately follows from a comparison of Equations (23) and (35). On moving with the fluid, $\mathrm{d}\vec{x}$ and $\vec{\omega}/\rho$ are transformed by the same matrix, J. Thus, if $\mathrm{d}\vec{x}$ and $\vec{\omega}$ are initially parallel, they stay parallel. A comoving material curve initially coinciding with a field line of $\vec{\omega}$ continues to do so.

5.2 Second Theorem: Conservation of the Flux of $\vec{\omega}$ Through Comoving Surfaces

For the flux of $\vec{\omega}/\rho$ through a comoving surface element, we have

$$\begin{aligned} \mathrm{d}\sigma^{\mathrm{t}}\frac{\vec{\omega}}{\rho} &= \det(J)\mathrm{d}\vec{\sigma}_0^{\mathrm{t}}J^{-1}J\frac{\vec{\omega_0}}{\rho_0} \\ &= \det(J)\mathrm{d}\vec{\sigma}_0^{\mathrm{t}}\frac{\vec{\omega_0}}{\rho_0} \\ &= \frac{1}{\rho}\mathrm{d}\vec{\sigma}_0^{\mathrm{t}}\vec{\omega_0}\,, \end{aligned} \tag{36}$$

which, on multiplying by ρ, gives

$$\mathrm{d}\sigma^{\mathrm{t}}\vec{\omega} = \mathrm{d}\vec{\sigma}_0^{\mathrm{t}}\vec{\omega_0}\,, \tag{37}$$

what was to be proven.

6 Magnetohydrodynamic Analogy: The Frozen-in Magnetic Field

Now suppose the fluid to be electrically conducting. It no longer needs to be barotropic. Also, no assumptions are made about viscosity and external forces.

We consider the case of non-relativistic ideal magnetohydrodynamics (MHD):
(1) The fluid velocity is much smaller than the speed of light, $|\vec{v}| \ll c$. In this limit, Maxwell's displacement current can be neglected compared to the conduction current.
(2) The electrical conductivity σ is infinitely high. Then, the magnetic diffusivity $1/(\mu_0\sigma)$ vanishes (analogous to the vanishing of the vortex diffusion in an inviscid fluid).

The magnetic field (more precisely: the magnetic induction) $\vec{B}$ then obeys

$$\frac{\partial\vec{B}}{\partial t} = \nabla\times(\vec{v}\times\vec{B})\,, \quad \nabla\vec{B} = 0\,. \tag{38}$$

From two equations for the vorticity $\vec{\omega}$ that are fully analogous to Eq. (38), namely, the vorticity equation, Eq. (8), and $\nabla\vec{\omega} = 0$, we have, additionally using the equation of mass conservation, Eq. 4, derived the two vortex theorems, whereby no explicit use was made of the relation $\vec{\omega} = \nabla\times\vec{v}$; merely the solenoidality of $\vec{\omega}$ was used. By analogy with the vortex theorems, we can therefore conclude:
(i) A material curve coinciding with a magnetic field line retains this property; "the magnetic field lines are conserved". This property immediately carries over to magnetic flux tubes.
(ii) The flux of $\vec{B}$ through a surface comoving with fluid is conserved.

The results (i) and (ii) are jointly referred to as Alfvén's theorem and the magnetic field is said to be "frozen into the fluid".