Norbert Seehafer, Institut für Physik und Astronomie, Universität Potsdam

The Navier-Stokes Equation in Cylindrical Coordinates r, ϕ, z

- Without the volume viscosity ("second viscosity")
- Without external forces
- In dimensionless form
- v_r, v_ϕ and v_z are the components of the flow velocity
- p is the pressure
- Re is the Reynolds number

$$\frac{\partial v_r}{\partial t} + (\vec{v}\cdot\nabla)\,v_r - \frac{v_\phi^2}{r} = -\frac{\partial p}{\partial r} + \mathrm{Re}^{-1}\left(\Delta v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\phi}{\partial \phi}\right) \tag{1}$$

$$\frac{\partial v_\phi}{\partial t} + (\vec{v}\cdot\nabla)\,v_\phi + \frac{v_r v_\phi}{r} = -\frac{1}{r}\frac{\partial p}{\partial \phi} + \mathrm{Re}^{-1}\left(\Delta v_\phi + \frac{2}{r^2}\frac{\partial v_r}{\partial \phi} - \frac{v_\phi}{r^2}\right) \tag{2}$$

$$\frac{\partial v_z}{\partial t} + (\vec{v}\cdot\nabla)\,v_z = -\frac{\partial p}{\partial z} + \mathrm{Re}^{-1}\Delta v_z \tag{3}$$

with

$$\vec{v}\cdot\nabla = v_r\frac{\partial}{\partial r} + \frac{v_\phi}{r}\frac{\partial}{\partial \phi} + v_z\frac{\partial}{\partial z} \tag{4}$$

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \tag{5}$$

The divergence of the flow velocity becomes:

$$\nabla\cdot\vec{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} \tag{6}$$

It should be noted that in Eq. (1)

$$\underbrace{\left[(\vec{v}\cdot\nabla)\,\vec{v}\right]_r}_{=(\vec{v}\cdot\nabla)\,v_r - \frac{v_\phi^2}{2}} \neq (\vec{v}\cdot\nabla)\,v_r \tag{7}$$

and in Eq. (2)

$$\underbrace{\left[\Delta\vec{v}\right]_\phi}_{=\Delta v_\phi + \frac{2}{r^2}\frac{\partial v_r}{\partial \phi} - \frac{v_\phi}{r^2}} \neq \Delta v_\phi \tag{8}$$