The Boussinesq Equations for Rotating Convection in a Spherical Shell

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We consider an incompressible fluid in a sperical shell with outer radius R_o and inner radius R_i, rotating with constant angular velocity $\boldsymbol{\Omega} = \Omega \boldsymbol{e}_z$ about the z axis ($\boldsymbol{e}_z$ is the unit vector in the z direction). The temperature T is fixed to the value T_o at radius R_o and to the value $T_o + \delta T$ at radius R_i. Using the Oberbeck-Boussinesq approximation, the governing equations for the fluid velocity $\boldsymbol{v}$ and temperature in the co-rotating frame read as follows:

$$\begin{aligned} \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} &= -2\Omega\, \boldsymbol{e}_z \times \boldsymbol{v} - \Omega^2 \boldsymbol{e}_z \times (\boldsymbol{e}_z \times \boldsymbol{r}) - \frac{1}{\rho_o}\nabla p + \nu \nabla^2 \boldsymbol{v} \\ &\quad + [1 - \alpha(T - T_o)]\, \boldsymbol{g} \qquad (1) \\ \frac{\partial T}{\partial t} + \boldsymbol{v}\cdot\nabla T &= \kappa \nabla^2 T \qquad (2) \\ \nabla\cdot\boldsymbol{v} &= 0 \qquad (3) \end{aligned}$$

$\boldsymbol{r}$ is the position vector, ρ_o the homogeneous mass density at temperature T_o, [1] p the pressure, ν the kinematic viscosity, α the thermal expansion coefficient, $\boldsymbol{g}$ the gravitational acceleration, and κ the thermal diffusivity. The first and second terms on the right-hand side of Eq. (1) give the Coriolis and centrifugal accelerations, respectively. Here, in particular, the effect of a temperature dependent mass density on the centrifugal force has been neglected. Employing the fact that for constant $\boldsymbol{\Omega}$ the centrifugal acceleration is a gradient,

$$-\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) = \nabla\left(\frac{1}{2}(\boldsymbol{\Omega}\times\boldsymbol{r})^2\right) = \nabla\left(\frac{1}{2}\Omega^2 \boldsymbol{r}_\perp^2\right), \quad \boldsymbol{r}_\perp = \boldsymbol{r} - \left(\boldsymbol{r}\cdot\frac{\boldsymbol{\Omega}}{\Omega}\right)\frac{\boldsymbol{\Omega}}{\Omega}, \qquad (4)$$

the centrifugal term in Eq. (1) will be included in the pressure term in the following. Furthermore, as $\boldsymbol{g}$ is always a potential field, $(1 + \alpha T_o)\,\boldsymbol{g}$ may be included in the

[1] T_o is the reference temperature and ρ_o the associated reference mass density for the Oberbeck-Boussinesq approximation; T_o and ρ_o may be, but need not be the values of temperature and mass desity at radius R_o

pressure term as well, so that Eq. (1) becomes

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -2\Omega\, \boldsymbol{e}_z \times \boldsymbol{v} - \frac{1}{\rho_o}\nabla p + \nu \nabla^2 \boldsymbol{v} - \alpha T \boldsymbol{g}\,. \tag{5}$$

Let the fluid shell be part of a full sphere with homogeneous mass density ρ_o. Then

$$\boldsymbol{g} = -\frac{4}{3}\pi\gamma\rho_o \boldsymbol{r}\,, \tag{6}$$

where γ is the gravitational constant. Eq. (6) may also be written as

$$\boldsymbol{g} = -\frac{g_o}{R_o}\boldsymbol{r}\,, \tag{7}$$

where g_o is the absolute value of the gravitational acceleration at radius R_o.

We now normalize as follows:

$$\boldsymbol{r}\,/D \to \boldsymbol{r}\,, \quad t \Big/ \frac{D^2}{\nu} \to t\,, \quad \boldsymbol{v} \Big/ \frac{\nu}{D} \to \boldsymbol{v}\,, \quad p\,/\rho_o \nu \Omega \to p\,, \quad T\,/\delta T \to T\,, \tag{8}$$

where $D = R_o - R_i$ is the gap size. The resulting non-dimensional equations read

$$\mathrm{E}\left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} - \nabla^2 \boldsymbol{v}\right) = -2\, \boldsymbol{e}_z \times \boldsymbol{v} - \nabla p + \mathrm{Ra}\, T \frac{\boldsymbol{r}}{R_o}\,, \tag{9}$$

$$\frac{\partial T}{\partial t} + \boldsymbol{v} \cdot \nabla T = \frac{1}{\mathrm{Pr}} \nabla^2 T\,, \tag{10}$$

$$\nabla \cdot \boldsymbol{v} = 0\,, \tag{11}$$

where

$$\mathrm{E} = \frac{\nu}{D^2 \Omega} \tag{12}$$

is the Ekman number,

$$\mathrm{Ra} = \frac{\alpha\, \delta T g_o\, D}{\Omega \nu} \tag{13}$$

is a modified Rayleigh number, and

$$\mathrm{Pr} = \frac{\nu}{\kappa} \tag{14}$$

is the Prandtl number. Normalizations of this kind were, e.g., used by Olson and Glatzmaier (1995), Christensen et al. (1998), and Christensen et al. (2001). A discussion of different definitions of the Rayleigh number, in the context of geodynamo simulation, was given by Kono and Roberts (2001).

Alternative non-dimensional form of the equations with homogeneous boundary conditions for the temperature

In the time-independent conductive basic state with the fluid at rest, Eqs. (9) and (10) become

$$0 = -\nabla p_c + \mathrm{Ra}\, T_c \,\frac{\boldsymbol{r}}{R_o}\,, \tag{15}$$

$$0 = \nabla^2 T_c \,, \tag{16}$$

where p_c and T_c denote pressure and temperature in the conductive state. The solution to Eq. (16) satisfying the boundary conditions

$$T_c(R_o) = T_o \,, \quad T_c(R_o - 1) = T_o + 1 \tag{17}$$

is

$$T_c = \frac{R_o(R_o-1)}{r} + T_o - R_o + 1 \,. \tag{18}$$

With T_c as given by Eq. (18), ∇p_c is fixed by Eq. (15) (and thus p_c is fixed up to an irrelevant constant). With the additional variable transformations

$$p - p_c \to p \,, \quad T - T_c \to \Theta \,, \tag{19}$$

and using Eqs. (9), (10), (15), (16), and (18), the Navier-Stokes and heat-conduction equations take the forms

$$\mathrm{E}\left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \nabla^2\boldsymbol{v}\right) = -2\,\boldsymbol{e}_z \times \boldsymbol{v} - \nabla p + \mathrm{Ra}\,\Theta\frac{\boldsymbol{r}}{R_o} \,, \tag{20}$$

$$\frac{\partial \Theta}{\partial t} + \boldsymbol{v}\cdot\nabla\Theta = \frac{1}{\mathrm{Pr}}\nabla^2\Theta + \frac{R_o(R_o-1)}{r^2} v_r \,. \tag{21}$$

References

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Appendices

A Transition to a more commonly used normalization

On dividing Eq. (9) by E and re-normalizing the pressure according to

$$\frac{p}{\mathrm{E}} \to p \tag{22}$$

(so that $\rho_0 \nu^2/D^2$ becomes the pressure unit), we obtain

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} - \nabla^2 \boldsymbol{v} = -\sqrt{\mathrm{Ta}}\, \boldsymbol{e}_z \times \boldsymbol{v} - \nabla p + \frac{1}{\mathrm{Pr}} \widetilde{\mathrm{Ra}}\, T \frac{\boldsymbol{r}}{R_o}\,, \tag{23}$$

where

$$\mathrm{Ta} = \frac{4}{\mathrm{E}^2} \tag{24}$$

is the Taylor number as most commonly defined (i.e., with a factor of 2 included in $\sqrt{\mathrm{Ta}}$) and

$$\widetilde{\mathrm{Ra}} = \frac{\alpha\, \delta T\, g_o\, D^3}{\kappa\, \nu} = \frac{\mathrm{Pr}}{\mathrm{E}}\, \mathrm{Ra} \tag{25}$$

is the conventional Rayleigh number.

B Taking into account centrifugal buoyancy

Allowing for a temperature dependence of the mass density in the centrifugal term in the same way as already done in the gravitational term, the dimensional Navier-Stokes equation takes the form

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -2\,\boldsymbol{\Omega} \times \boldsymbol{v} + \alpha\, T\, \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) - \frac{1}{\rho_o} \nabla p + \nu \nabla^2 \boldsymbol{v} + \alpha\, T\, g_o \frac{\boldsymbol{r}}{R_o}\,, \tag{26}$$

where the gradient part $-(1 + \alpha T_o)\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})$ of the centrifugal term $-[1 - \alpha(T - T_o)]\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})$ has been included in the pressure term. On normalizing as given by Eq. (8), the non-dimensional Navier-Stokes equation becomes

$$\mathrm{E}\left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} - \nabla^2 \boldsymbol{v}\right) = -2\,\boldsymbol{e}_z \times \boldsymbol{v} + \mathrm{Ra}\,\mathrm{Fr}\, T\, \boldsymbol{e}_z \times (\boldsymbol{e}_z \times \boldsymbol{r}) - \nabla p + \mathrm{Ra}\, T \frac{\boldsymbol{r}}{R_o}\,, \tag{27}$$

where

$$\mathrm{Fr} = \frac{D\Omega^2}{g_o} \tag{28}$$

is the (rotational) Froude number.