Additions to the article “Determination of constant α force-free solar magnetic fields from magnetograph data” by N. Seehafer, in Solar Physics, Vol. 58, pp. 215–223 (1978)

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The following additions to the article by Seehafer (1978), “Determination of constant α force-free solar magnetic fields from magnetograph data” in Solar Physics, Vol. 58, pp. 215-223 (1978), are made:

(i) The proof that solutions of the Helmholtz equation in exterior domains are not square-integrable is given in detail.

(ii) The proof of a theorem on the energy content of magnetic fields that are force-free in all space by Molodenskii (1969) is schown to be incorrect.

(iii) A minor correction is made to Eq. (6) in Seehafer (1978).

1 The Helmholtz equation in exterior domains

Applying a standard procedure, we decompose the Laplace operator in spherical polar coordinates into radial and angular parts according to

$$\nabla^2 = \Delta = \frac{1}{r^2}(\Delta_r + \Delta_\Omega), \tag{1}$$

with

$$\Delta_r = \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right), \quad \Delta_\Omega = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}, \tag{2}$$

where Δ_Ω acts on the unit sphere Ω centered at the origin. The solutions P to the scalar Helmholtz equation,

$$\Delta P + \alpha^2 P = 0, \tag{3}$$

are then expanded in a spherical-harmonic series as

$$P(r,\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{j=-n}^{n} R_{nj}(r) Y_n^{(j)}(\theta,\varphi). \tag{4}$$

The spherical harmonics are eigenfunctions of Δ_Ω, namely,

$$\Delta_\Omega Y_n^{(j)}(\theta,\varphi) = -n(n+1) Y_n^{(j)}(\theta,\varphi). \tag{5}$$

From Eqs. (1), (4) and (5) we get

$$\begin{aligned}
\Delta P(r,\theta,\varphi) &= \sum_{n=0}^{\infty}\sum_{j=-n}^{n} \frac{1}{r^2}(\Delta_r + \Delta_\Omega) R_{nj}(r) Y_n^{(j)}(\theta,\varphi) \\
&= \sum_{n=0}^{\infty}\sum_{j=-n}^{n} \frac{1}{r^2}\left\{[\Delta_r R_{nj}(r)] Y_n^{(j)}(\theta,\varphi) + R_{nj}(r)\Delta_\Omega Y_n^{(j)}(\theta,\varphi)\right\} \\
&= \sum_{n=0}^{\infty}\sum_{j=-n}^{n} \frac{1}{r^2}\left\{\Delta_r R_{nj}(r) - n(n+1) R_{nj}(r)\right\} Y_n^{(j)}(\theta,\varphi).
\end{aligned} \tag{6}$$

The Helmholtz equation, Eq. (3), then yields an equation for the coefficients of the series expansion, independent of j:

$$\frac{1}{r^2}\Delta_r R_{nj}(r) + \left(\alpha^2 - \frac{n(n+1)}{r^2}\right) R_{nj}(r) = 0 \tag{7}$$

or, equivalently,

$$\frac{\mathrm{d}^2 R_{nj}(r)}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}R_{nj}(r)}{\mathrm{d}r} + \left(\alpha^2 - \frac{n(n+1)}{r^2}\right) R_{nj}(r) = 0 \tag{8}$$

The solutions to Eq. (8) can be written as

$$R_{nj}(r) = \frac{A_{nj}}{\sqrt{r}} J_{n+1/2}(r) + \frac{B_{nj}}{\sqrt{r}} N_{n+1/2}(r), \tag{9}$$

where $J_{n+1/2}(r)$ and $N_{n+1/2}(r)$ are the Bessel and Neumann functions, respectively, of order $n+\frac{1}{2}$, and A_{nj} and B_{nj} are constants (see, e.g., Jackson, 1999), or, by using Hankel functions of the first and second kind,

$$H_\nu^{(1)}(r) = J_\nu(r) + \mathrm{i}N_\nu(r), \quad H_\nu^{(2)}(r) = J_\nu(r) + \mathrm{i}N_\nu(r), \tag{10}$$

in the form

$$R_{nj}(r) = \frac{\tilde{A}_{nj}}{\sqrt{r}} H_{n+1/2}^{(1)}(r) + \frac{\tilde{B}_{nj}}{\sqrt{r}} H_{n+1/2}^{(2)}(r). \tag{11}$$

Assuming the spherical harmonics to be normalized on Ω, that is,

$$\int_\Omega Y_n^{(j)}(\theta,\varphi)\, Y_k^{(l)}(\theta,\varphi)\, \mathrm{d}\Omega = \delta_{nk}\delta_{jl}, \tag{12}$$

Eq. (4) gives

$$\int_{r>r_o} [P(r,\theta,\varphi)]^2 \mathrm{d}V = \int_{r_o}^{\infty} \left\{ \sum_{n=0}^{\infty} \sum_{j=-n}^{n} [R_{nj}(r)]^2 \right\} r^2 \mathrm{d}r = \sum_{n=0}^{\infty} \sum_{j=-n}^{n} \int_{r_o}^{\infty} [R_{nj}(r)]^2 \, r^2 \mathrm{d}r \, . \tag{13}$$

In the limit $r \to \infty$, the Bessel, Neumann and Hankel functions decay as $1/\sqrt{r}$,

$$J_\nu(r) = \mathcal{O}\left(\frac{1}{\sqrt{r}}\right), \quad N_\nu(r) = \mathcal{O}\left(\frac{1}{\sqrt{r}}\right), \quad \left|H_\nu^{(1,2)}(r)\right| = \mathcal{O}\left(\frac{1}{\sqrt{r}}\right), \tag{14}$$

thus

$$R_{nj}(r) = \mathcal{O}\left(\frac{1}{r}\right), \quad P(r,\theta,\varphi) = \mathcal{O}\left(\frac{1}{r}\right), \quad [P(r,\theta,\varphi)]^2 = \mathcal{O}\left(\frac{1}{r^2}\right). \tag{15}$$

This decay as $r \to \infty$ is too slow for the solutions of the Helmholtz equation to be square-integrable in exterior domains, in particular the exterior of a sphere, since the integral of a positive function over such a domain diverges if it decays as $r^{-\gamma}$ with $\gamma \leq 3$. This property of the Helmholtz equation was first proved by Rellich (1943).

Physical implication Applying the operator $\nabla\times$ to the equation

$$\nabla \times \boldsymbol{B} = \alpha \boldsymbol{B}, \quad \alpha = \text{constant}, \tag{16}$$

we obtain

$$\nabla \times \nabla \times \boldsymbol{B} = \alpha^2 \boldsymbol{B}, \quad \Delta \boldsymbol{B} + \alpha^2 \boldsymbol{B} = 0, \tag{17}$$

where we have employed the vector identity $\nabla \times \nabla \times \boldsymbol{B} = \nabla(\nabla \cdot \boldsymbol{B}) - \Delta \boldsymbol{B}$ and $\nabla \cdot \boldsymbol{B} = 0$. That is, $\boldsymbol{B}$ satisfies the vector Helmholtz equation. Thus, in a system of Cartesian coordinates x, y, z, all three field components, B_x, B_y and B_z, satisfy the scalar Helmholtz equation. As a consequence, the integral of $\boldsymbol{B}^2$ over an exterior domain diverges—a constant-alpha force-free magnetic field cannot have a finite energy content in such a domain (except for the case $\alpha = 0$).

2 Virial theorem and the theorem of Molodenskii (1969) on magnetic fields that are force-free in all space

From a virial theorem, Molodenskii (1969) obtained the result that the energy of a force-free magnetic field in a spherical volume increases at least proportionally to the radial coordinate r as $r \to \infty$. [1] While this is in accordance with our result for the constant-α fields, Molodenskii's result applies to all force-free fields. Can that be true?

Molodenskii's line of argument is similar to the one used for proving the well-known non-existence theorem, which states that no force-free magnetic fields exist for which

[1]When publishing the paper Solar Phys. Vol. 58, pp. 215–223 (1978), the author was not aware of the paper by Molodenskii (1969).

the currents are confined to a finite volume (see, e.g., Chandrasekhar, 1961; Roberts, 1967; Moffatt, 1978). The derivation relies on the fact that the magnetic force $\boldsymbol{\mathcal{F}}$ can be expressed as the divergence of the magnetic stress tensor T:

$$T_{ij} = \frac{1}{\mu_0} B_i B_j - \frac{\boldsymbol{B}^2}{2\mu_0}\delta_{ij} \tag{18}$$

Specifically,

$$\boldsymbol{\mathcal{F}} = \boldsymbol{j} \times \boldsymbol{B} = \frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} = \operatorname{div} T, \tag{19}$$

with, using the summation convention,

$$[\operatorname{div} T]_i = \partial_j T_{ij}. \tag{20}$$

If the currents are localized in a sphere of radius r_o with surface S, and the magnetic field is force-free ($\boldsymbol{\mathcal{F}} = \boldsymbol{0}$), Gauss's theorem yields

$$\begin{aligned}
\int_{r=r_\mathrm{o}} (T\boldsymbol{r})\,\mathrm{d}\boldsymbol{S} &= \int_{r<r_\mathrm{o}} \nabla \cdot (T\boldsymbol{r})\,\mathrm{d}V \\
&= \int_{r<r_\mathrm{o}} \partial_i (T_{ij} r_j)\,\mathrm{d}V \\
&= \int_{r<r_\mathrm{o}} [(\partial_i T_{ij}) r_j + T_{ij}\partial_i r_j]\,\mathrm{d}V \\
&= \int_{r<r_\mathrm{o}} T_{ii}\,\mathrm{d}V \\
&= -\frac{1}{2\mu_0}\int_{r<r_\mathrm{o}} \boldsymbol{B}^2\,\mathrm{d}V,
\end{aligned} \tag{21}$$

where we have used $\partial_i T_{ij} = 0$ and $\partial_i r_j = \delta_{ij}$. If we let $r_\mathrm{o} \to \infty$, the surface integral on the left-hand side of Eq. (21) tends to zero since $\boldsymbol{B}$ falls off at least as r^{-3} (the dipole field); thus, $\boldsymbol{B} \equiv \boldsymbol{0}$. The proof holds if the volume is a spherical shell with inner radius r_i and outer radius r_o because, as presupposed for the theorem, the field in the enclosed sphere ($r < r_\mathrm{i}$) is current-free and the full sphere ($r < r_\mathrm{o}$) is force-free. If, however, the field in the enclosed sphere is not force-free, an integral over the inner boundary at radius r_i must be added to the left-hand side of Eq. (21). This term remains finite as $r_\mathrm{o} \to \infty$, allowing force-free magnetic field with localized currents to exist.

Then the question arises whether there are magnetic fields that are force-free throughout all space. The answer is yes: The constant-α fields provide examples of such fields. Namely, the solutions P to the Helmholtz equation, as given by Eqs. (4) and (9), can be extended from the exterior to the interior of a sphere. The second term on the right-hand side of Eq. (9) must then be excluded, since the Neumann functions diverge as their argument approaches zero. The first term, involving the Bessel functions $J_{n+1/2}$, however, is analytic (infinitely differentiable) at the origin.

Up to this point, we have not explicitly used the form of T given by Eq. (18); we relied only on the vanishing of its divergence and the proportionality of its components to products of the magnetic field components. Employing the explicit form of T, the integrand of the surface integral on the left-hand side of Eq. (21) can be written in index notation as

$$T_{ij}r_j = \frac{1}{\mu_0}B_iB_jr_j - \frac{\boldsymbol{B}^2}{2\mu_0}\delta_{ij}r_j = \frac{\boldsymbol{B}\cdot\boldsymbol{r}}{\mu_0}B_i - \frac{\boldsymbol{B}^2}{2\mu_0}r_i, \tag{22}$$

giving

$$T_{ij}r_j\mathrm{d}S_i = \frac{\boldsymbol{B}\cdot\boldsymbol{r}}{\mu_0}B_i\mathrm{d}S_i - \frac{\boldsymbol{B}^2}{2\mu_0}r_i\mathrm{d}S_i \tag{23}$$

and, with $\mathrm{d}S_i = (r_i/r)\mathrm{d}S$,

$$T_{ij}r_j\mathrm{d}S_i = \frac{\boldsymbol{B}\cdot\boldsymbol{r}}{\mu_0 r}B_ir_i\mathrm{d}S - \frac{\boldsymbol{B}^2}{2\mu_0 r}r_ir_i\mathrm{d}S, \tag{24}$$

taking the form

$$(T\boldsymbol{r})\,\mathrm{d}\boldsymbol{S} = \frac{B_r^2}{\mu_0}r\mathrm{d}S - \frac{\boldsymbol{B}^2}{2\mu_0}r\mathrm{d}S \tag{25}$$

in vector notation. Eqs. (21) and (25) result in

$$\begin{aligned}\int_\Omega \frac{B_r^2}{\mu_0}r_\mathrm{o}^3\mathrm{d}\Omega - \int_\Omega \frac{\boldsymbol{B}^2}{2\mu_0}r_\mathrm{o}^3\mathrm{d}\Omega &= -\frac{1}{2\mu_0}\int_{r<r_\mathrm{o}}\boldsymbol{B}^2\,\mathrm{d}V \\ &= -\frac{1}{2\mu_0}\int_0^{r_\mathrm{o}}\int_\Omega \boldsymbol{B}^2r^2\mathrm{d}\Omega\,\mathrm{d}r \end{aligned}\tag{26}$$

and, with B_t denoting the modulus of the tangential component of $\boldsymbol{B}$ on the sphere with radius r_o,

$$\frac{r_\mathrm{o}^3}{2\mu_0}\int_\Omega (B_r^2 - B_t^2)\mathrm{d}\Omega = -\frac{1}{2\mu_0}\int_0^{r_\mathrm{o}}\left(r^2\int_\Omega \boldsymbol{B}^2\,\mathrm{d}\Omega\right)\mathrm{d}r\,. \tag{27}$$

Molodenskii (1969) derives an equation analogous to Eq. (27) (see his Eq. (4) in Gaussian units). However, his version erroneously contains a factor of r_o^2 instead of r_o^3 in front of the surface integral on the left-hand side and lacks the minus sign on the right-hand side. He defines (in our notation with SI units)

$$W(r_\mathrm{o}) = \frac{1}{2\mu_0}\int_0^{r_\mathrm{o}}\left(r^2\int_\Omega \boldsymbol{B}^2\,\mathrm{d}\Omega\right)\mathrm{d}r, \tag{28}$$

$$E(r_\mathrm{o}) = \frac{r_\mathrm{o}^2}{\mu_0}\int_\Omega (B_t^2)\mathrm{d}\Omega\,, \tag{29}$$

then

$$\frac{\mathrm{d}W}{\mathrm{d}r_\mathrm{o}} = \frac{r_\mathrm{o}^2}{2\mu_0}\int_\Omega \boldsymbol{B}^2\,\mathrm{d}\Omega\,, \tag{30}$$

and obtains

$$W = \left(\frac{\mathrm{d}W}{\mathrm{d}r_\mathrm{o}} - E\right)r_\mathrm{o}\,, \tag{31}$$

(his Eq. (5)), although a correct derivation from his (incorrect) Eq. (4) would have resulted in

$$W = \frac{\mathrm{d}W}{\mathrm{d}r_\mathrm{o}} - E\,. \tag{32}$$

Differentiation of Eq. (31) yields

$$\frac{\mathrm{d}W}{\mathrm{d}r_\mathrm{o}} = \frac{\mathrm{d}^2W}{\mathrm{d}r_\mathrm{o}^2}r_\mathrm{o} + \frac{\mathrm{d}W}{\mathrm{d}r_\mathrm{o}} - \frac{\mathrm{d}E}{\mathrm{d}r_\mathrm{o}}r_\mathrm{o} - E\,, \tag{33}$$

$$E = \left(\frac{\mathrm{d}^2W}{\mathrm{d}r_\mathrm{o}^2} - \frac{\mathrm{d}E}{\mathrm{d}r_\mathrm{o}}\right) r_\mathrm{o} \tag{34}$$

and, since E and r_o are positive,

$$\frac{\mathrm{d}}{\mathrm{d}r_\mathrm{o}}\left(\frac{\mathrm{d}W}{\mathrm{d}r_\mathrm{o}} - E\right) = \frac{\mathrm{d}}{\mathrm{d}r_\mathrm{o}}\left(\frac{r_\mathrm{o}^2}{2\mu_0}\int_\Omega (B_r^2 - B_t^2)\,\mathrm{d}\Omega\right) \geq 0\,, \tag{35}$$

that is, $(\mathrm{d}W/\mathrm{d}r_\mathrm{o} - E)$ would increase with r_o and, because of Eq. (31), be positive. It could then still be bounded. In such a case, it would approach some finite positive limit C. Since $\boldsymbol{B}^2 \geq B_r^2 - B_t^2$, it follows that $\mathrm{d}W/\mathrm{d}r_\mathrm{o}$ (given by Eq. (30)) would be bounded from below by C; it could then be either bounded or unbounded from above as $r_\mathrm{o} \to \infty$. Note that $\mathrm{d}W/\mathrm{d}r_\mathrm{o}$ is the integrand of the radial integral in Eq. (28), representing the magnetic energy $W(r_\mathrm{o})$ within the sphere $r < r_\mathrm{o}$. Consequently, for large r_o, the magnetic energy in the sphere would grow at the same rate as r_o if $\mathrm{d}W/\mathrm{d}r_\mathrm{o}$ were bounded from above; otherwise it would grow even faster. This reasoning, used (in part implicitly) by Molodenskii (1969), is unfortunately based on an incorrect equation. The correct equations do not support conclusions of this kind.

3 A correction to Eq. (6) in Seehafer (1978)

Finally, a correction to Eq. (6) in Seehafer (1978) (the equation is mentioned, but not used in the paper): The first of the conditions there must obviously be changed from $P = o(r^{-1})$ to $P = \mathcal{O}(r^{-1})$, cf. Eq. (15) above. [2]

The second condition, which can also be written as

$$\lim_{r\to\infty} r\left(\frac{\partial P}{\partial r} + \mathrm{i}\alpha P\right) = 0, \tag{36}$$

is the Sommerfeld radiation condition. It is important in studies of spherical waves. If a solution $\psi(\boldsymbol{r}, t)$ of the scalar, source-free wave equation is Fourier-analyzed in time as

$$\psi(\boldsymbol{r}, t) = \int_{-\infty}^{\infty} \hat{\psi}(\boldsymbol{r}, \omega) e^{\mathrm{i}\omega t}\mathrm{d}\omega, \tag{37}$$

the coefficient function satisfies the scalar Helmholtz equation,

$$(\Delta + k^2)\hat{\psi}(\boldsymbol{r}, \omega) = 0, \tag{38}$$

[2]The error occured as follows: In the submitted manuscript (typewritten at that time), the conditions were correct, with $\mathcal{O}(r^{-1})$ in the first and $o(r^{-1})$ in the second one. In the proofs, $\mathcal{O}(r^{-1})$ was mistakenly used in both conditions. The author changed $\mathcal{O}(r^{-1})$ to $o(r^{-1})$ in the second condition. In the printed paper, however, $o(r^{-1})$ was used in both conditions, mistakenly including the first one.

with $k^2 = \omega^2/c^2$. The radiation condition, Eq. (36), with $P \to \psi$ and $\alpha \to k$, selects outgoing waves and excludes incoming ones. This corresponds to the convention in engineering, where the time dependence $e^{\mathrm{i}\omega t}$ is used, see Eq. (37). In physics, the convention $e^{-\mathrm{i}\omega t}$ is more common. Then $\mathrm{i}\alpha$ ($\mathrm{i}k$) in Eq. (36) must be replaced by $-\mathrm{i}\alpha$ ($-\mathrm{i}k$). With this latter choice, the term involving the Hankel functions of the first kind $H^{(1)}_{n+1/2}(r)$ in Eq. (11) represents outgoing waves, while the term involving the Hankel functions of the second kind $H^{(2)}_{n+1/2}(r)$ represents incoming waves.

References

Chandrasekhar, S. (1961). *Hydrodynamic and Hydromagnetic Stability*. Oxford: Clarendon Press.

Jackson, J. D. (1999). *Classical Electrodynamics*. 3rd ed. New York: Wiley.

Moffatt, H. K. (1978). *Magnetic Field Generation in Electrically Conducting Fluids*. Cambridge, UK: Cambridge University Press.

Molodenskii, M. M. (1969). “Integral properties of force-free fields”. In: *Soviet Astronomy – AJ* 12, pp. 585–588. Original article in: *Astronomicheskii Zhurnal*, Vol. 45, No. 4, pp. 732–737, 1968.

Rellich, F. (1943). “Über das asymptotische Verhalten der Lösungen von $\Delta u + \lambda u = 0$ in unendlichen Gebieten”. In: *Jahresbericht der Deutschen Mathematiker-Vereinigung* 53, pp. 57–65. In German. English title: “On the asymptotic behavior of solutions of $\Delta u + \lambda u = 0$ in infinite domains”.

Roberts, P. H. (1967). *An Introduction to Magnetohydrodynamics*. London: Longmans.

Seehafer, N. (1978). “Determination of constant α force-free solar magnetic fields from magnetograph data”. In: *Solar Phys.* 58, pp. 215–223.