DETERMINATION OF CONSTANT α FORCE-FREE SOLAR MAGNETIC FIELDS FROM MAGNETOGRAPH DATA

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Abstract. At first it is shown that a magnetic field being force-free, i.e. satisfying $\nabla\times\mathbf{B}=\alpha\mathbf{B}$, with $\alpha=\text{constant}$ ($\alpha\neq 0$) in the whole exterior of the Sun cannot have a finite energy content and cannot be determined uniquely from only one magnetic field component given at the photosphere. Then the boundary value problem for a semi-infinite column of arbitrary cross section is solved by a Green's function method.

1. Introduction

Magnetic fields play an important role in almost all events in the solar atmosphere. Therefore there is a strong need for information about the magnetic flux density vector throughout the atmosphere. At present, however, reliable and detailed information about magnetic fields is available only for the photospheric level, where the inverse Zeeman effect in Fraunhofer lines is observable, most observations being restricted to the line-of-sight component only. The implications about magnetic fields that can be drawn from chromospheric and coronal observations, such as Hα fibrils and loops seen in EUV and X-ray lines, are very limited. Therefore, the measured line-of-sight component of the photospheric field must be extrapolated into the field vector in the higher layers. The mostly used way of doing this is to assume that the magnetic field is current-free (potential) or, more general, force-free above the photosphere, and then to solve the arising equation with the measured photospheric magnetic field distribution providing the boundary condition. The use of force-free fields is justified by the dominance of the magnetic field and its stability in the chromosphere and lower corona (Sturrock and Woodbury, 1967). As a consequence, if appreciable currents are present, these must be aligned with the magnetic field, since otherwise the resulting Lorentz forces could not be balanced by nonmagnetic forces. Therefore, neglecting displacement currents,

$$\nabla\times\mathbf{B}=\alpha\mathbf{B}\,, \tag{1}$$

where α is, in general, a scalar function of position (and time). The practical application is confined to the case of $\alpha=\text{const.}$, α being treated as a parameter that must be determined by comparison of the computed magnetic field configuration with chromospheric or coronal features that can serve as tracers of magnetic field topology, for example loops seen in various lines. For varying α one needs information about α throughout the considered volume. $\alpha = 0$ corresponds to the current-free case.

There are three extensively applicated extrapolation procedures (Levine, 1975):

(a) The Schmidt procedure (Schmidt, 1964):
The potential field above a limited photospheric region is computed. Implicitly, a Neumann boundary value problem for the upper half space is solved, assuming that the magnetic field component normal to the boundary plane vanishes everywhere in this plane outside the region covered by the measurement (magnetogram).

(b) The procedure of Altschuler and Newkirk (1969):
The potential field in the full volume between the photosphere and a surface at some radial distance R_1 (about $2.5R_\odot$) is computed. The assumption that the field becomes radial, thereby simulating the effects of the solar wind, provides the boundary condition at $r = R_1$.

(c) The procedure of Nakagawa and Raadu (1972):
The constant α force-free magnetic field above a rectangular photospheric region is computed. Implicitly the input data are two dimensionally periodically extrapolated into the whole magnetogram plane (Seehafer, 1975).

(a) and (c) start from incomplete data. The need for specifying the boundary conditions on vertical planes comprising the considered volume (semi-infinite column above magnetogram area) is eliminated by extending the volume into a half space.

In this paper at first the case of the magnetic field being force-free with $\alpha =$ constant in the full volume outside the Sun (above photosphere) is considered. Then the boundary value problem for a semi-infinite column of arbitrary cross section is solved by a Green's function method.

2. The Exterior Boundary Value Problem for the Sphere

With $\alpha =$ constant (1) is equivalent to

$$\mathbf{B} = \alpha \mathbf{r} \times \nabla P + \nabla \times (\mathbf{r} \times \nabla P), \tag{2}$$

where $\mathbf{r}$ is a constant vector (Nakagawa and Raadu, 1972) or the radius vector (Rädler, 1974), and the scalar function P satisfies the Helmholtz equation

$$(\nabla^2 + \alpha^2)P = 0. \tag{3}$$

By taking the divergence of (1) it can be seen that div $\mathbf{B} = 0$ is automatically fulfilled (except for $\alpha = 0$).

There is a difference between the Laplace equation (which must be solved in the case of the potential field) and the Helmholtz equation: If a solution of the Laplace equation vanishes at the boundary surface S of a finite region and at infinity, it vanishes identically outside S, while under similar conditions a solution of the

Helmholtz equation does not necessarily vanish (Müller, 1957). For example,

$$f_1(\mathbf{r}) = \frac{\sin \alpha r}{r} \tag{4}$$

and

$$f_2(\mathbf{r}) = \frac{\cos \alpha r}{r}, \tag{5}$$

where $\mathbf{r}$ denotes the radius vector and $r = |\mathbf{r}|$, are solutions of the Helmholtz equation that vanish at infinity. One can always find a linear combination of f_1 and f_2 that vanishes at some distance $r = R$ (and at infinity).

An exterior boundary value problem with a unique solution can be posed by means of an extra condition at infinity (Müller, 1957), namely the Sommerfeld radiation condition

$$P = o(r^{-1}), \qquad \frac{\partial P}{\partial r} + i\alpha P = o(r^{-1}). \tag{6}$$

Let us consider the exterior boundary value problem for the sphere in detail: P can be represented by a harmonic expansion

$$P(r, \theta, \varphi) = \sum_{n=0}^{\infty} \sum_{j=-n}^{n} R_{nj}(r) Y_n^{(j)}(\theta, \varphi), \tag{7}$$

where $Y_n^{(j)}$ denote spherical harmonics of degree n and order j. Then, the coefficients $R_{nj}(r)$ must satisfy the equation

$$R''_{nj} + \frac{2}{r} R'_{nj} + \left(\alpha^2 - \frac{n(n+1)}{r^2}\right) R_{nj} = 0. \tag{8}$$

The general solution of (8) can be written in the form

$$R_{nj}(r) = A_{nj} \frac{1}{\sqrt{r}} H^{(1)}_{n+1/2}(\alpha r) + B_{nj} \frac{1}{\sqrt{r}} H^{(2)}_{n+1/2}(\alpha r), \tag{9}$$

where $H^{(1)}_{n+1/2}$ and $H^{(2)}_{n+1/2}$ denote the Hankel functions of the first and second kind, respectively, of the order $n + \frac{1}{2}$, and A_{nj}, B_{nj} are arbitrary constants.

Clearly, if only one magnetic field component is given at $r = R$ (photosphere), it is impossible to determine both A_{nj} and B_{nj}. Nakagawa (1973) gives a solution of the boundary value problem with only one set of constants to be determined. Obviously, he does not use the general solution of (8), although claiming to do this.

To define a unique boundary value problem, an additional condition is needed. The Sommerfeld radiation condition (6), which is used in studying wave propagation, is not adequate to the problem. An adequate condition would be that the magnetic field outside the sun had a finite energy content.

The magnetic energy content M within the volume of analysis is given by

$$M = \frac{1}{8\pi} \iiint_V |\mathbf{B}|^2 \, \mathrm{d}V. \tag{10}$$

From (2) (with $\mathbf{r}$ being a constant vector, for convenience), it can easily be verified that in a system of rectangular Cartesian coordinates x, y, z not only the function P but also the magnetic field components B_x, B_y, B_z satisfy the Helmholtz Equation (3).

Now, one can show (Rellich, 1943) that, if u is a function that satisfies the Helmholtz Equation (3) with $\alpha \neq 0$ for $r > R$, the function

$$f(R_1) = \iiint_{R<r<R_1} |u|^2 \, \mathrm{d}V \tag{11}$$

tends to infinity as $R_1 \to \infty$.

Thus, we have the remarkable result: A magnetic field being force-free with $\alpha = \text{constant}$ everywhere outside the sun cannot have a finite energy content (except for $\alpha = 0$).

3. Green's Function Method for a Semi-Infinite Column

In the following we use Cartesian coordinates x, y, z and the representation (2) with $\mathbf{r} = (0, 0, 1)$. Let the domain of analysis be given by

$$0 \leq x \leq L_x, \qquad 0 \leq y \leq L_y, \qquad 0 \leq z < \infty,$$

where $z = 0$ defines the plane of magnetograph observation. If the Green's function $G(\mathbf{r}, \mathbf{r}')$ for the Helmholtz Equation (3) in the considered volume is known, the values of any solution P can be determined from the values of P at the boundary by means of the representation

$$P(\mathbf{r}') = -\iint_S P(\mathbf{r}) \frac{\partial G}{\partial n} \, \mathrm{d}S, \tag{12}$$

where S denotes the surface enclosing the considered volume. n is the local exterior normal on S.

For constructing the Green's function we consider the inhomogeneous equation

$$(\nabla^2 + \alpha^2) Q = -F \tag{13}$$

with the boundary condition that Q vanishes at S.

By means of the Green's function, a solution Q of (13) can be represented in the form

$$Q(\mathbf{r}') = \iiint_V G(\mathbf{r}, \mathbf{r}') F(\mathbf{r}) \, \mathrm{d}V. \tag{14}$$

Let $\psi_{mn}(x, y)$ denote the normalized eigenfunctions of the two-dimensional

Laplacian $\Delta_\perp$ in the region $0 \leq x \leq L_x$, $0 \leq y \leq L_y$, i.e. the solutions of the equation

$$\Delta_\perp \psi_{mn} + \lambda_{mn}\psi_{mn} = 0 \tag{15}$$

that vanish at $x = 0$, L_x and $y = 0$, L_y.

They are given by

$$\psi_{mn}(x, y) = \sqrt{\frac{4}{L_x L_y}} \sin\left(\frac{\pi m}{L_x} x\right) \sin\left(\frac{\pi n}{L_y} y\right). \tag{16}$$

The corresponding eigenvalues are

$$\lambda_{mn} = \pi^2\left(\frac{m^2}{L_x^2} + \frac{n^2}{L_y^2}\right). \tag{17}$$

Using (15) and the expansions

$$Q(x, y, z) = \sum_{m,n=1}^{\infty} Q_{mn}(z)\psi_{mn}(x, y), \tag{18}$$

$$F(x, y, z) = \sum_{m,n=1}^{\infty} F_{mn}(z)\psi_{mn}(x, y), \tag{19}$$

we get from (13)

$$Q''_{mn} + (\alpha^2 - \lambda_{mn})Q_{mn} = -F_{mn}. \tag{20}$$

In verifying this note that the expansion (18) can be differentiated twice term by term with respect to x and y. Consider, for example, the dependence on x: Because of the boundary condition, Q can be considered as an odd function in the interval $-L_x \leq x \leq L_x$ and, therefore, $\partial Q/\partial x$ as an even function. The values of both Q and $\partial Q/\partial x$ at $x = -L_x$ and $x = L_x$ are equal. Therefore, the Fourier expansions representing them can be differentiated term by term.

The solution of (20) that vanishes both at $z = 0$ and as $z \to \infty$ is given by

$$Q_{mn}(z) = \frac{1}{2r_{mn}} \int_0^\infty F_{mn}(t)\{e^{-r_{mn}|z-t|} - e^{-r_{mn}|z+t|}\}\,\mathrm{d}t, \tag{21}$$

where

$$r_{mn} = \sqrt{\lambda_{mn} - \alpha^2}.$$

This can easily be shown by splitting the integral on the right-hand side of (21) according to $\int_0^\infty = \int_0^z + \int_z^\infty$ and considering the two summands of the integrand separately.

Because of

$$F_{mn}(t) = \int_0^{L_x} \int_0^{L_y} F(x, y, t)\psi_{mn}(x, y)\,\mathrm{d}x\,\mathrm{d}y, \tag{22}$$

(21) becomes

$$Q_{mn}(z)=\frac{1}{2r_{mn}}\iiint\limits_V F(x,y,t)\psi_{mn}(x,y)\times$$
$$\times\{e^{-r_{mn}|z-t|}-e^{-r_{mn}|z+t|}\}\,dx\,dy\,dt. \tag{23}$$

Putting this expression into (18) and interchanging summation and integration, we get a representation of Q according to (14) with

$$G(\mathbf{r},\mathbf{r}')=\sum_{m,n=1}^{\infty}\frac{\psi_{mn}(x,y)\psi_{mn}(x',y')}{2r_{mn}}\{e^{-r_{mn}|z-z'|}-e^{-r_{mn}|z+z'|}\}. \tag{24}$$

(24) gives the Green's function for the Helmholtz equation in a semi-infinite column of rectangular cross section. If the cross section has another form, the ψ_{mn} must be replaced by the eigenfunctions of the two-dimensional Laplacian in a region of corresponding form. For example, considering a semi-infinite cylinder Bessel functions must be used.

The use of the Green's function method in applicating special boundary conditions shall be illustrated in an example:

The magnetograph observation provides the boundary condition in terms of the vertical magnetic field component B_z at $z=0$. Since B_z satisfies the Helmholtz Equation (3), it can be determined uniquely by specifying its values at the vertical planes enclosing the volume. Then, after determining P from the equation

$$B_z=-\alpha^2P-\frac{\partial^2P}{\partial z^2}, \tag{25}$$

which follows from (2), B_x and B_y can be obtained from (2).

Let B_z vanish at the vertical planes $x=0$, $x=L_x$, $y=0$, $y=L_y$. It is assumed that the values of $|B_z|$ at the boundary are small compared with its values at the inner parts of the cross section of the column. Then, from (2), (12), and (24) we get

$$B_x=\sum_{m,n=1}^{\infty}\frac{C_{mn}}{\lambda_{mn}}e^{-r_{mn}z}\left\{\alpha\frac{\pi n}{L_y}\sin\left(\frac{\pi mx}{L_x}\right)\cos\left(\frac{\pi ny}{L_y}\right)-\right.$$
$$\left.-r_{mn}\frac{\pi m}{L_x}\cos\left(\frac{\pi mx}{L_x}\right)\sin\left(\frac{\pi ny}{L_y}\right)\right\}, \tag{26}$$

$$B_y=-\sum_{m,n=1}^{\infty}\frac{C_{mn}}{\lambda_{mn}}e^{-r_{mn}z}\left\{\alpha\frac{\pi m}{L_x}\cos\left(\frac{\pi mx}{L_x}\right)\sin\left(\frac{\pi ny}{L_y}\right)+\right.$$
$$\left.+r_{mn}\frac{\pi n}{L_y}\sin\left(\frac{\pi mx}{L_x}\right)\cos\left(\frac{\pi ny}{L_y}\right)\right\}, \tag{27}$$

$$B_z=\sum_{m,n=1}^{\infty}C_{mn}e^{-r_{mn}z}\sin\left(\frac{\pi mx}{L_x}\right)\sin\left(\frac{\pi ny}{L_y}\right), \tag{28}$$

where the C_{mn} are defined by the expansion (28) at $z=0$. (For the practical application the magnetogram must be changed or completed in such a way that the values at the boundary vanish, for example by adding an artificial boundary.)

The representation (26), (27), (28) has been used by in studying the large sunspot group of August 1972 (Seehafer and Staude, 1977, 1978). It does not require the net magnetic flux through the magnetogram area to be zero, as do the Schmidt procedure and the procedure of Nakagawa and Raadu.

There is a close similarity of the Equations (26)–(28) with the Equations (18)–(20) of Nakagawa and Raadu (1972). The relation of the two procedures shall be considered in detail:

The solution (26), (27), (28) is periodic, as the solution of Nakagawa and Raadu, but with the periods $2L_x$ and $2L_y$, whereas the solution of Nakagawa and Raadu has the periods L_x and L_y. One can get the solution (26), (27), (28) by extrapolating the magnetogram, which covers the domain $0 \leq x \leq L_x$, $0 \leq y \leq L_y$, into the domain $-L_x \leq x \leq L_x$, $-L_y \leq y \leq L_y$ according to

$$B_z(-x, y, 0) = -B_z(x, y, 0), \tag{29}$$

$$B_z(x, -y, 0) = -B_z(x, y, 0), \tag{30}$$

and then applicating the formulae of Nakagawa and Raadu to the extrapolated magnetogram. The solution of Nakagawa and Raadu is got in the same way if the extrapolation of the magnetogram into the domain $-L_x \leq x \leq L_x$, $-L_y \leq y \leq L_y$ is carried out according to (instead of (29), (30))

$$B_z(-x, y, 0) = B_z(x, y, 0), \tag{31}$$

$$B_z(x, -y, 0) = B_z(x, y, 0). \tag{32}$$

The use of (29), (30) instead of (31), (32) ensures that the net magnetic flux through the (original) magnetogram area is balanced.

As to the conditions at the (vertical part of the) boundary corresponding to the two solutions, the present solution is more restrictive in that it requires the vertical magnetic field component to vanish at the boundary, whereas the solution of Nakagawa and Raadu only requires that

$$B_z(0, y, z) = B_z(L_x, y, z) \tag{33}$$

and

$$B_z(x, 0, z) = B_z(x, L_y, z). \tag{34}$$

But, since it is assumed to be twice differentiable term by term with respect to x and y, the solution of Nakagawa and Raadu requires additionally (Seehafer, 1975)

$$\left.\frac{\partial B_z(x, y, z)}{\partial x}\right|_{x=0} = \left.\frac{\partial B_z(x, y, z)}{\partial x}\right|_{x=L_x}, \tag{35}$$

$$\left.\frac{\partial B_z(x, y, z)}{\partial y}\right|_{y=0} = \left.\frac{\partial B_z(x, y, z)}{\partial y}\right|_{y=L_y}, \tag{36}$$

which the present solution does not.

4. Discussion

In this paper it is shown (Section 2) that a magnetic field that is force-free with $\alpha = \text{constant}$ ($\alpha \neq 0$) in the whole volume outside the sun cannot have a finite energy content and that such a field cannot be determined uniquely from only one magnetic field component given at the photosphere. Therefore, the extension of a global scale constant α force-free magnetic field to infinity neither has a physical meaning nor provides a mathematically unique boundary value problem. The use of global scale constant α force-free magnetic fields must be confined to the consideration of finite volumes. A generalization of the method of Altschuler and Newkirk (1969), in which the potential field between the photosphere and a surface at some radial distance (where the field is assumed to become radial) is computed, to cases of $\alpha \neq 0$ seems most reasonable. However, since the curl of a radial field has no radial component, the way of matching to the outer solar wind dominated regions must be appropriately changed then.

The start from global scale magnetograms in extrapolating photospheric magnetic fields has the advantage (compared with the start from magnetograms covering limited regions) that no reference (in terms of special boundary conditions) to fields surrounding the magnetogram area is needed. Its disadvantage is the necessity to assume that the fields are steady for a period of about one month, which is needed to get a complete magnetic map of the photosphere. Thus, the extrapolation starting from magnetograms covering limited regions is also of interest. Moreover, this is the case so far as special physical assumptions, such as the constancy of α, are more questionable on a global scale.

If one computes the constant α force-free magnetic field in a semi-infinite column above a limited photospheric region, it is inadequate to eliminate the need for boundary conditions at the vertical parts of the surface bounding the considered volume by extending the volume into either the whole exterior of a sphere or a half space, except for $\alpha = 0$. These extensions do not provide unique boundary value problems. Chiu and Hilton (1977) have shown that the boundary value problem in a half space, using only the normal field component at the boundary as boundary values, is non-unique. It should be clarified if the result that a constant α force-free magnetic field cannot have a finite energy content is also valid for the half space (a mathematically non-trivial problem).

In the case of the extrapolation starting from magnetograms covering limited regions the boundary conditions must be treated carefully: Comparisons of field line calculations for the same magnetogram region using the Schmidt procedure and using the procedure of Nakagawa and Raadu can show drastic differences in the configuration of field lines, these discrepancies being due to the different treatment of the boundary conditions in the two schemes (Levine, 1975). The Green's function method given in Section 3 renders it possible to determine the constant α force-free magnetic field (including the case $\alpha = 0$) above a limited photospheric region from boundary conditions at the surface enclosing the actually considered volume. Different boundary conditions, especially such considered as realistic for physical reasons, can be imposed, the presently used extrapolation methods being included as special cases in the derived general scheme. The practical use of the scheme has been illustrated in deriving an extrapolation procedure which uses boundary conditions somewhat different from those of the procedure of Nakagawa and Raadu and, therefore, does not require that net magnetic flux through the magnetogram area to be zero.

References

Altschuler, M. D. and Newkirk, G., Jr.: 1969, *Solar Phys.* **9**, 131.
Chiu, Y. T. and Hilton, H. H.: 1977, *Astrophys. J.* **212**, 873.
Levine, R. H.: 1975, *Solar Phys.* **44**, 365.
Müller, C.: 1957, *Grundprobleme der mathematischen Theorie elektromagnetischer Schwingungen*, Springer-Verlag, Berlin-Göttingen-Heidelberg, p. 81.
Nakagawa, Y.: 1973, *Astron. Astrophys.* **27**, 95.
Nakagawa, Y. and Raadu, M. A.: 1972, *Solar Phys.* **25**, 127.
Rädler, K.-H.: 1974, *Astron. Nachr.* **295**, 73.
Rellich, F.: 1943, *Jber. Deutsch. Math. Verein.* **53**, 57.
Schmidt, H. U.: 1964, in *AAS-NASA Symposium on the Physics of Solar Flares*, NASA SP-50, p. 107.
Seehafer, N.: 1975, *Astron. Nachr.* **296**, 177.
Seehafer, N. and Staude, J.: 1977, *Publ. Debrecen Heliophys. Obs.* (in press).
Seehafer, N. and Staude, J.: 1978, in preparation for *Astron. Nachr.*
Sturrock, P. A. and Woodbury, E. T.: 1967, in *Plasma Astrophysics*, 39th Enrico Fermi School, Academic Press, New York and London, p. 155.