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Current Helicity and the Turbulent Electromotive Force.

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Abstract. – In a magnetofluid, magnetic and velocity fluctuations $\boldsymbol{B}'$ and $\boldsymbol{v}'$ can generate a mean electromotive force along the mean magnetic field, which has been termed the alpha-effect. In the present paper the mathematical relationship between the alpha-effect and the mean current helicity of the fluctuations, $\langle \boldsymbol{B}' \cdot \mathrm{curl}\, \boldsymbol{B}' \rangle$, hitherto proven for the case that i) either the magnetic or the velocity fluctuations are statistically homogeneous and stationary, ii) the first-order smoothing approximation (FOSA) is valid, and iii) the mean flow, $\langle \boldsymbol{v} \rangle$, vanishes, is rederived, assuming the magnetic fluctuations to be stationary and the coupled magnetic-electric fluctuations to be homogeneous, but without using FOSA and allowing for a non-vanishing mean flow.

In order to explain the origin of the cosmical magnetic fields, the theory of the turbulent dynamo has been developed [1-3]. The central mechanism in this theory is the alpha-effect, namely the generation of a mean electromotive force (e.m.f) along a mean, or large-scale, magnetic field by turbulently fluctuating, or small-scale, parts of velocity and magnetic field. The effect has also been invoked to understand the plasma behaviour in fusion experiments [4].

It has been found that kinetic and magnetic helicities can enable a turbulent dynamo effect. With $\boldsymbol{v}$, $\boldsymbol{B}$ and $\boldsymbol{A}$ denoting fluid velocity, magnetic field and a magnetic vector potential, the densities per unit volume of kinetic, magnetic and current helicity are defined by

$$H_\mathrm{K} = \boldsymbol{v} \cdot \mathrm{curl}\, \boldsymbol{v}; \qquad H_\mathrm{M} = \boldsymbol{A} \cdot \boldsymbol{B}; \qquad H_\mathrm{C} = \boldsymbol{B} \cdot \mathrm{curl}\, \boldsymbol{B}. \tag{1}$$

Let the evolution of the magnetic field be described by the induction equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \mathrm{curl}\,(\boldsymbol{v} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}, \tag{2}$$

with a constant magnetic diffusivity $\eta = (\mu_0 \sigma)^{-1}$, σ denoting the electrical conductivity. If, as usual in mean-field electrodynamics, velocity and magnetic fields are split up into mean and fluctuating parts according to

$$\boldsymbol{v} = \langle \boldsymbol{v} \rangle + \boldsymbol{v}', \qquad \boldsymbol{B} = \langle \boldsymbol{B} \rangle + \boldsymbol{B}', \tag{3}$$

with angular brackets denoting averages and primes the corresponding residuals, then eq. (2) can be separated into two coupled equations for the time evolution of $\langle \boldsymbol{B} \rangle$ and $\boldsymbol{B}'$, namely

$$\frac{\partial \langle \boldsymbol{B} \rangle}{\partial t} = \operatorname{curl}(\langle \boldsymbol{v} \rangle \times \langle \boldsymbol{B} \rangle) + \eta \nabla^2 \langle \boldsymbol{B} \rangle + \operatorname{curl} \mathcal{E} , \tag{4}$$

$$\frac{\partial \boldsymbol{B}'}{\partial t} = \operatorname{curl}(\boldsymbol{v}' \times \langle \boldsymbol{B} \rangle) + \operatorname{curl}(\langle \boldsymbol{v} \rangle \times \boldsymbol{B}') + \eta \nabla^2 \boldsymbol{B}' + \operatorname{curl} \boldsymbol{G} , \tag{5}$$

where

$$\mathcal{E} = \langle \boldsymbol{v}' \times \boldsymbol{B}' \rangle \tag{6}$$

and

$$\boldsymbol{G} = \boldsymbol{v}' \times \boldsymbol{B}' - \langle \boldsymbol{v}' \times \boldsymbol{B}' \rangle . \tag{7}$$

$\mathcal{E}$ is the mean e.m.f. caused by the fluctuations.

Formally the averages can be understood as ensemble averages. They can, however, physically meaningfully be defined as space, time, or space-time averages if the turbulence has the two-scale property, *i.e.* if the characteristic scales (space, time, or space-time) of the fluctuations are much smaller than those of the mean fields, so that proper means can be obtained by averaging over intermediate scales.

Usually $\mathcal{E}$ is evaluated by assuming the mean velocity to vanish,

$$\langle \boldsymbol{v} \rangle = 0 , \tag{8}$$

and applying the first-order smoothing approximation (FOSA), which consists in neglecting $\operatorname{curl} \boldsymbol{G}$ in eq. (5). Then eq. (5) simplifies to

$$\frac{\partial \boldsymbol{B}'}{\partial t} = \operatorname{curl}(\boldsymbol{v}' \times \langle \boldsymbol{B} \rangle) + \eta \nabla^2 \boldsymbol{B}' , \tag{9}$$

which with $\langle \boldsymbol{B} \rangle$ taken as a constant (justified in the case of a space-time two-scale turbulence with $\langle \ldots \rangle$ defined as space-time average) can be easily solved by using Fourier-transformation techniques, allowing for an expression of $\boldsymbol{B}'$ and, therefore, also of $\mathcal{E}$ in terms of $\boldsymbol{v}'$. One thus arrives at relations revealing a connection between the alpha-effect and the kinetic helicity H'_{K} of $\boldsymbol{v}'$.

As found by Keinigs [5], who used eq. (9) to express $\mathcal{E}$ in terms of $\boldsymbol{B}'$, the alpha-effect is, however, more directly related to *current* helicity than to kinetic helicity, namely by the equation

$$\alpha \equiv \frac{\mathcal{E} \cdot \langle \boldsymbol{B} \rangle}{\langle \boldsymbol{B} \rangle^2} = - \frac{\eta}{\langle \boldsymbol{B} \rangle^2} \langle \boldsymbol{B}' \cdot \operatorname{curl} \boldsymbol{B}' \rangle . \tag{10}$$

In deriving this, Keinigs assumed the *magnetic* fluctuations to be statistically stationary, homogeneous and symmetric about the direction of $\langle \boldsymbol{B} \rangle$. Then $\mathcal{E}$ and $\langle \boldsymbol{B} \rangle$ are parallel. For the case that the *velocity* fluctuations represent a turbulence that is stationary and homogeneous, but not necessarily rotationally symmetric about any axis (or even fully isotropic), and the scalar α has consequently to be replaced by a tensorial quantity, Rädler and Seehafer [6] derived the relation

$$a_{ij}\langle B_i\rangle\langle B_j\rangle = -\eta\langle \boldsymbol{B}'\cdot\mathrm{curl}\,\boldsymbol{B}'\rangle , \tag{11}$$

where Cartesian coordinates x_1, x_2, x_3 have been used and the tensor a_{ij} is defined by

$$\mathcal{E}_i = a_{ij}\langle B_j\rangle . \tag{12}$$

Equation (10) has been written in such a way that also this slightly more general case is covered. In the following the relation (10) is rederived without applying FOSA and without using eq. (8):

Let the magnetic fluctuations be statistically stationary and the coupled magnetic-electric fluctuations be statistically homogeneous. To formulate the actually needed assumption on the spatial symmetry more precisely: the mixed magnetic-electric two-point correlation tensor $Q_{ij}(\boldsymbol{x}, \boldsymbol{r}) = \langle B_i'(\boldsymbol{x})E_j'(\boldsymbol{x}+\boldsymbol{r})\rangle$, $\boldsymbol{E}'$ denoting the fluctuating part of the electric field $\boldsymbol{E}$, is supposed to be independent of $\boldsymbol{x}$.

Our rederivation of eq. (10) will be essentially based on a consideration of the mean value of the fluctuating part of the magnetic helicity, $\langle \boldsymbol{A}'\cdot\boldsymbol{B}'\rangle$. Now, a statistically homogeneous field is by definition of infinite spatial extent, so that it is difficult to define a proper vector potential for $\boldsymbol{B}'$. This difficulty can be overcome by working with relative magnetic helicity, H_{R}, as defined by Berger and Field [7]. H_{R} is not a helicity density, as *e.g.* H_{M}, but measures the total, *i.e.* volume-integrated helicity of a magnetic field in a (in general finite) volume V, namely by comparing it to the current-free field with the same normal component on ∂V. More explicitly: the field $\boldsymbol{B}$ given in V is extended into the exterior $\hat{V}$ of V by the current-free field in $\hat{V}$ with the same normal component on ∂V as the given field. From the total magnetic helicity of the entire field defined in all space one then subtracts that of the field $\boldsymbol{B}_{\mathrm{p}}$ which is current-free on both sides of ∂V and whose normal component on ∂V is again that of the given field ($\boldsymbol{B}_{\mathrm{p}}$ is thus generated by a current-sheet on ∂V). H_{R} depends only on the field $\boldsymbol{B}$ in V.

Let V be a sphere cut out of our infinitely extended field. The rate of change of H_{R} in V is given by [7]

$$\frac{\mathrm{d}H_{\mathrm{R}}(V)}{\mathrm{d}t} = -2\int_V \boldsymbol{E}\cdot\boldsymbol{B}\,\mathrm{d}V + 2\int_{\partial V}(\boldsymbol{A}_{\mathrm{p}}\times\boldsymbol{E})\cdot\boldsymbol{n}\,\mathrm{d}S . \tag{13}$$

Here $\boldsymbol{n}$ is the exterior unit normal on ∂V and $\boldsymbol{A}_{\mathrm{p}}$ the Coulomb vector potential (satisfying $\mathrm{div}\,\boldsymbol{A}_{\mathrm{p}} = 0$) of $\boldsymbol{B}_{\mathrm{p}}$. The volume integral on the right-hand side of eq. (13) is due to the internal dissipation of magnetic helicity, the surface integral describes the flow of magnetic helicity through ∂V.

For the fluctuating part H_{R}' of H_{R} one finds a relation fully analogous to eq. (13), which after averaging reads

$$\frac{\mathrm{d}\langle H_{\mathrm{R}}'(V)\rangle}{\mathrm{d}t} = -2\int_V \langle \boldsymbol{E}'\cdot\boldsymbol{B}'\rangle\,\mathrm{d}V + 2\int_{\partial V}\langle \boldsymbol{A}_p'\times\boldsymbol{E}'\rangle\cdot\boldsymbol{n}\,\mathrm{d}S . \tag{14}$$

A coupling between fluctuating and mean fields does not occur here, since only linear equations are used to define H_{R} and to calculate its rate of change. So eq. (14) is derived from Maxwell's equations $\mathrm{div}\,\boldsymbol{B}' = 0$ and $\partial\boldsymbol{B}'/\partial t = -\mathrm{curl}\,\boldsymbol{E}'$ and $\boldsymbol{B}' = \mathrm{curl}\,\boldsymbol{A}'$.

Because of the assumed statistical stationarity of the magnetic fluctuations, the left-hand side of eq. (14) must vanish. The surface integral on the right-hand side vanishes as a consequence of the spatial homogeneity of the correlation between magnetic and electric fluctuations: A'_p can be represented in the form

$$A'_p(x) = \frac{\mu_0}{4\pi}\int_{\partial V} \frac{J'_p(x_a)}{|x - x_a|}\, dS_a\,, \tag{15}$$

where J'_p is the density of the sheet current on ∂V that generates B'_p. It is given by the jump $\Delta B'_p$ of the tangential component of B'_p across ∂V,

$$\mu_0 J'_p(x_a) = n_a \times \Delta B'_p(x_a)\,. \tag{16}$$

By using appropriate Green's functions for the potential-field problems in V and its complement $\widehat{V}$, $\Delta B'_p$ in turn admits of a representation

$$\Delta B'_p(x_a) = \int_{\partial V} (B'(x_b)\cdot n_b)\, G(x_a, x_b)\, dS_b\,. \tag{17}$$

The detailed form of the vector-valued function $G(x_a, x_b)$ is not needed here. We shall only use the fact that A'_p at a given position on ∂V is obtained as the superposition of the influences of the values of $B'\cdot n$ at all other points. Let r, θ, ϕ be spherical polar coordinates and consider that part of the helicity flux at the north pole, $\theta = 0$, of our sphere which is determined by the value of $B'\cdot n$ at the point (θ_0, ϕ_0). Compare this with the helicity flux at $(\theta_0 + \pi, \phi_0)$ due to the influence of the south pole, $\theta = \pi$. The vector r from (θ_0, ϕ_0) to the north pole is equal to that from the south pole to $(\theta_0 + \pi, \phi_0)$. So the two-point correlation between fluctuating magnetic and electric fields is the same for both point pairs and the considered parts of the helicity fluxes at the north pole and at the point $(\theta_0 + \pi, \phi_0)$ cancel one another.

The latter can perhaps be seen more clearly from the fact that the relevant quantities for the second point pair are obtained from those of the first one by a mirror reflexion in the plane through the centre of the sphere and perpendicular to the radius vector of the point $(\theta_0/2, \phi_0)$. The helicity flow as a pseudoscalar quantity must change sign under such a reflexion.

Since, in this way, for any contribution to the helicity flow another one just cancelling it can be found, the total flow of relative magnetic helicity through ∂V must vanish.

Letting now $V \to 0$, we arrive at

$$\langle E'\cdot B'\rangle = 0\,. \tag{18}$$

Next we derive an expression for $\langle E'\cdot B'\rangle$ following from Ohm's law. Using $\mu_0 j = \operatorname{curl} B$ and eq. (6), the unaveraged and averaged forms of Ohm's law can be written as

$$E = \frac{\langle j\rangle + j'}{\sigma} - (\langle v\rangle + v') \times (\langle B\rangle + B') =$$

$$= \eta \operatorname{curl}\langle B\rangle + \eta \operatorname{curl} B' - \langle v\rangle \times \langle B\rangle - \langle v\rangle \times B' - v' \times \langle B\rangle - v' \times B' \tag{19}$$

and

$$\langle E\rangle = \eta \operatorname{curl}\langle B\rangle - \langle v\rangle \times \langle B\rangle - \mathcal{E}\,. \tag{20}$$

For their difference we then find

$$E' = E - \langle E\rangle = \eta \operatorname{curl} B' - \langle v\rangle \times B' - v' \times \langle B\rangle - v' \times B' + \mathcal{E}\,, \tag{21}$$

which, by scalar multiplication with $\boldsymbol{B}'$ and subsequent averaging, gives

$$\boldsymbol{E}' \cdot \boldsymbol{B}' = \eta \operatorname{curl} \boldsymbol{B}' \cdot \boldsymbol{B}' + (\boldsymbol{v}' \times \boldsymbol{B}')\langle \boldsymbol{B} \rangle + \mathcal{E} \cdot \boldsymbol{B}' \tag{22}$$

and

$$\langle \boldsymbol{E}' \cdot \boldsymbol{B}' \rangle = \eta \langle \operatorname{curl} \boldsymbol{B}' \cdot \boldsymbol{B}' \rangle + \mathcal{E} \cdot \langle \boldsymbol{B} \rangle . \tag{23}$$

Note that eq. (23) has been derived without any assumption except for Ohm's law in the form $\boldsymbol{j} = \sigma(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$. It shows that, as already noticed by Keinigs and Gerwin [8] for the case of $\langle \boldsymbol{v} \rangle = 0$, eq. (10) is equivalent to eq. (18). We have, however, derived eq. (18) independently and, therefore, achieved an independent derivation of eq. (10).

Most remarkable in our rederivation seems to be the exemption from FOSA.

Our presuppositions concerning the temporal and spatial symmetry of the turbulence differ slightly from those of Keinigs [5], who assumed the magnetic-magnetic (instead of the magnetic-electric) correlations to be independent of position.

Nothing has been assumed about the mean velocity. So we have in particular dispensed with the usual assumption $\langle \boldsymbol{v} \rangle = \boldsymbol{0}$. If $\langle \boldsymbol{v} \rangle$ is uniform, then one can change to a coordinate system moving with velocity $\langle \boldsymbol{v} \rangle$ with respect to the original one. A non-uniform mean velocity, on the other hand, has represented a considerable difficulty in traditional dynamo theory and has, for the calculation of the turbulent e.m.f., been taken into account at most approximately (see ref. [1], Chapter 8).

The component of the turbulent e.m.f. in the direction of $\langle \boldsymbol{B} \rangle$, $\alpha \langle \boldsymbol{B} \rangle$, can lead to an increase as well as to a decrease of the energy contained in the mean magnetic field. In the first case it acts as a dynamo. Equation (10) can be used to distinguish between both possibilities [8, 6, 9]: to be able to pick up energy from the fluctuations, the mean magnetic field must possess a current helicity, $\langle \boldsymbol{B} \rangle \cdot \operatorname{curl} \langle \boldsymbol{B} \rangle$, whose sign is opposite to that of the mean current helicity of the fluctuations.

Additional Remark

It has been pointed out by an anonymous referee that, instead of working with relative magnetic helicity, one can alternatively define $\boldsymbol{A}'$ via the relation between the Fourier transforms (considered as generalised functions) of $\boldsymbol{A}'$ and $\boldsymbol{B}'$. Then eq. (18) is readily obtained using normal Coulomb gauge for $\boldsymbol{A}'$.

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