FORCE-FREE MAGNETIC FIELDS IN STATIC MEDIA

(Letter to the Editor)

N. SEEHAFER
Arbeitsgruppe "Nichtlineare Dynamik" der Max-Planck-Gesellschaft an der Universiät Potsdam, Am Neuen Palais, Potsdam, Germany

(Received: 24 November, 1992)

Abstract. In a static resistive medium, a force-free magnetic field must decay exponentially with time while preserving the form of its field lines. This has been proven by Jette (1970) in an extremely complex way. In the present paper a simpler proof of the same result is given.

1. Introduction

A force-free magnetic field is one that satisfies

$$\operatorname{curl} \mathbf{B} \times \mathbf{B} = 0 \tag{1}$$

or, equivalently,

$$\operatorname{curl} \mathbf{B} = \alpha \mathbf{B} \tag{2}$$

with some scalar function α. For a review on force-free fields see Boström (1973).

From Eq. (2) and

$$\operatorname{div} \mathbf{B} = 0, \tag{3}$$

it follows that

$$\operatorname{grad} \alpha \cdot \mathbf{B} = 0, \tag{4}$$

that is, the field lines lie in surfaces α = constant.

In a static medium the time evolution of $\mathbf{B}$ is governed by

$$\frac{\partial \mathbf{B}}{\partial t} = -\eta \operatorname{curl}^2 \mathbf{B}, \tag{5}$$

where η denotes the magnetic diffusivity, which is assumed to be a constant. From Eqs. (5) and (2) one finds the equation

$$\frac{1}{\eta}\frac{\partial \mathbf{B}}{\partial t} = -\operatorname{grad} \alpha \times \mathbf{B} - \alpha^2 \mathbf{B}, \tag{6}$$

which can be decomposed into two equations for the absolute value of $\mathbf{B}$ and the unit vector $\mathbf{b}$ in the direction of $\mathbf{B}$, respectively:

Astrophysics and Space Science **202:** 383–386, 1993.
© 1993 *Kluwer Academic Publishers. Printed in Belgium.*

$$\frac{1}{\eta}\frac{\partial \mathbf{B}^2}{\partial t} = -2\alpha^2\mathbf{B}^2, \tag{7}$$

$$\frac{1}{\eta}\frac{\partial \mathbf{b}}{\partial t} = -\operatorname{grad}\alpha \times \mathbf{b}. \tag{8}$$

Eq. (8) shows that for a spatially constant α the form of the field lines does not change with time. Then α is also independent of time since it is a function of the field line geometry alone. Namely, from Eq. (2) it follows that

$$\alpha = \frac{\mathbf{B}\cdot\operatorname{curl}\mathbf{B}}{\mathbf{B}^2} = \mathbf{b}\cdot\operatorname{curl}\mathbf{b}. \tag{9}$$

If α is a constant, the solution of the system of Eqs. (2) and (5) (Eq. (3) is automatically satisfied in this special case) takes the form

$$\mathbf{B} = \mathbf{B}_0\exp(-\eta\alpha^2 t), \tag{10}$$

where $\mathbf{B}_0$ is the field at time $t = 0$.

Now it has been shown by Jette (1970) that a force-free magnetic field in a resistive medium at rest can remain force-free in time only if α is constant in both space and time. For his proof he used two additional equations which are obtained by differentiating Eq. (1) (or Eq. (2), respectively) twice with respect to time.

Jette's proof, in which only local algebraic and differential operations are used, is extremely complex and hard to follow. Therefore Lo Surdo (1979) has given arguments to derive the same conclusion by considering the initial-boundary value problem for a force-free magnetic field in a finite volume. In the present paper once more the *local* problem is considered to give a simplified version of Jette's proof.

2. Proof of the Constancy of α

In the following we use notations $\mathbf{A}$ and β defined by

$$\mathbf{A} = \operatorname{curl}(\operatorname{grad}\alpha \times \mathbf{B}), \tag{11}$$

$$\beta = \frac{\mathbf{A}\cdot\mathbf{B}}{\mathbf{B}^2} = -\frac{1}{\eta}\frac{\partial\alpha}{\partial t}, \tag{12}$$

where the equivalence on the right of Eq. (12) is obtained by differentiating Eq. (9) with respect to time.

Differentiation of Eq. (1) with respect to time yields by using Eq. (5) the five equivalent equations

$$\operatorname{curl}^3\mathbf{B}\times\mathbf{B} = \operatorname{curl}^2\mathbf{B}\times\operatorname{curl}\mathbf{B}, \tag{13}$$

$$\operatorname{curl}^3 \mathbf{B} - \alpha \operatorname{curl}^2 \mathbf{B} = \beta \mathbf{B}, \tag{14}$$

$$\mathbf{A} \times \mathbf{B} = \alpha \mathbf{B}^2 \operatorname{grad} \alpha, \tag{15}$$

$$\mathbf{B} \times (\operatorname{grad} \alpha \cdot \nabla)\mathbf{B} = 0, \tag{16}$$

$$(\operatorname{grad} \alpha \cdot \nabla)\mathbf{b} = 0. \tag{17}$$

For the derivation of Eq. (15) the relation

$$\operatorname{curl}^2 \mathbf{B} = \operatorname{grad} \alpha \times \mathbf{B} + \alpha^2 \mathbf{B}, \tag{18}$$

which follows from Eq. (2), can be used. Eqs. (16) and (17) may be obtained by using the vector identity

$$\operatorname{curl}(\operatorname{grad} \alpha \times \mathbf{B}) = -2(\operatorname{grad} \alpha \cdot \nabla)\mathbf{B} - (\triangle \alpha)\mathbf{B} - \alpha(\operatorname{grad} \alpha \times \mathbf{B}). \tag{19}$$

They were first derived by Jette and Sreenivasan (1969) and show, as noted by these authors, that **b** is constant along the grad α-lines (this is particularly seen from Eq. (17)), so that each grad α-line lies in a plane, namely the plane perpendicular to its **b**.

The left of Eq. (14) is divergence-free, as is seen using Eqs.'(18) and (4), and so

$$\operatorname{grad} \beta \cdot \mathbf{B} = 0, \tag{20}$$

which is already obvious from Eq. (8). From Eqs. (17) and (8) we further see that besides α and β also $(\operatorname{grad} \alpha)^2$ is costant along the **b**-lines:

$$\begin{aligned} \mathbf{b} \cdot \operatorname{grad}(\operatorname{grad} \alpha)^2 &= 2\mathbf{b} \cdot (\operatorname{grad} \alpha \cdot \nabla) \operatorname{grad} \alpha \\ &= 2\mathbf{b} \cdot (\operatorname{grad} \alpha \cdot \nabla)(-\frac{1}{\eta}\mathbf{b} \times \frac{\partial \mathbf{b}}{\partial t}) \\ &= -\frac{2}{\eta}\mathbf{b} \cdot (\mathbf{b} \times (\operatorname{grad} \alpha \cdot \nabla)\frac{\partial \mathbf{b}}{\partial t}) = 0. \end{aligned} \tag{21}$$

By differentiating Eq. (13) with respect to time and using Eq. (5) we get as a further equation

$$\operatorname{curl}^5 \mathbf{B} \times \mathbf{B} + 2 \operatorname{curl}^3 \mathbf{B} \times \operatorname{curl}^2 \mathbf{B} + \operatorname{curl} \mathbf{B} \times \operatorname{curl}^4 \mathbf{B} = 0, \tag{22}$$

which can also be written as

$$\begin{aligned}&\operatorname{curl}^3(\operatorname{grad}\alpha\times\mathbf{B})\times\mathbf{B}-2(\operatorname{grad}(\operatorname{grad}\alpha)^2\times\mathbf{B})\times\mathbf{B}+\\&+\alpha\operatorname{curl}^2(\operatorname{grad}\alpha\times\mathbf{B})\times\mathbf{B}=(\mathbf{B}\cdot\operatorname{curl}(\operatorname{grad}\alpha\times\mathbf{B}))\operatorname{grad}\alpha.\end{aligned}\tag{23}$$

By using in particular Eq. (15) in the form

$$\mathbf{A}=\beta\mathbf{B}-\alpha(\operatorname{grad}\alpha\times\mathbf{B})$$

and the vector identity (19) with α replaced by β, Eq. (23) can be brought into the equivalent forms

$$2\mathbf{B}\times(\operatorname{grad}\beta\cdot\nabla)\mathbf{B}=-\mathbf{B}^2\operatorname{grad}(\operatorname{grad}\alpha)^2+\beta\mathbf{B}^2\operatorname{grad}\alpha,\tag{24}$$

$$2(\operatorname{grad}\beta\cdot\nabla)\mathbf{b}=\mathbf{b}\times\operatorname{grad}(\operatorname{grad}\alpha)^2-\beta\mathbf{b}\times\operatorname{grad}\alpha.\tag{25}$$

On the other hand, by differentiating Eq. (17) with respect to time we get

$$2(\operatorname{grad}\beta\cdot\nabla)\mathbf{b}=\mathbf{b}\times\operatorname{grad}(\operatorname{grad}\alpha)^2.\tag{26}$$

Comparing Eqs. (25) and (26) we see that

$$\beta(\mathbf{b}\times\operatorname{grad}\alpha)=0\tag{27}$$

and, where $\operatorname{grad}\alpha\neq 0$,

$$\beta=0.\tag{28}$$

That is, α and the $\operatorname{grad}\alpha$-lines are independent of time. Then by differentiating Eq. (9) with respect to time and using Eqs. (8), (11), and (12) we find

$$\operatorname{curl}\mathbf{b}\cdot\frac{\partial\mathbf{b}}{\partial t}=0\tag{29}$$

$$\left(\operatorname{grad}\frac{1}{|\mathbf{B}|}\times\mathbf{b}\right)\cdot\frac{\partial\mathbf{b}}{\partial t}=0,\tag{30}$$

$$\operatorname{grad}\alpha\cdot\operatorname{grad}\frac{1}{|\mathbf{B}|}=0,\tag{31}$$

which shows that $|\mathbf{B}|$ is costant along the $\operatorname{grad}\alpha$-lines. According to Eq. (7), at each spatial point $|\mathbf{B}|$ decreases exponentially with time. It can remain constant along a $\operatorname{grad}\alpha$-line only if the value of α is the same along this line, that is, if $\operatorname{grad}\alpha=0$.

References

Boström, R.: 1973, *Astrophys. Space Sci.* **22**, 353.
Jette, A.D.: 1970, *J. Math. Anal. Appl.* **29**, 109.
Jette, A.D. and Sreenivasan, S.R.: 1969, *Phys. Fluids* **12**, 2544.
Lo Surdo, C.: 1979, *J. Appl. Math. Phys.* **30**, 647.