Relations between Helicities in Mean-field Dynamo Models

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ABSTRACT

Within the framework of mean-field dynamo theory some results concerning the current helicities of fluctuating and mean magnetic fields are derived. Using the second order correlation approximation a relation between the current helicity of the fluctuating magnetic field and parameters determining the α-effect is established. On this basis it is shown that the energy stored in the mean magnetic field of an α^2-dynamo is prevented from decaying only if, at least in some region, the current helicities of the fluctuating and the mean magnetic fields have opposite signs and the modulus of the former exceeds that of the latter. Results of an analysis of magnetic field configurations in solar active regions are also presented and discussed with reference to current helicity and the α-effect.

1. INTRODUCTION

In investigations of dynamo processes which are believed to be responsible for the magnetic fields of cosmical bodies, the mean-field approach has proved useful (see, e.g., Krause & Rädler, 1980). In this approach the magnetic flux density, $\mathbf{B}$, in an electrically conducting fluid as well as the velocity, $\mathbf{u}$, of the fluid motions are understood as superpositions of mean parts, $<\mathbf{B}>$ and $<\mathbf{u}>$, which are defined by a proper averaging procedure, and fluctuating parts, $\mathbf{B}'$ and $\mathbf{u}'$. The mean magnetic flux density $<\mathbf{B}>$ inside the fluid is governed by the equation

$$\eta\Delta <\mathbf{B}> + \text{curl } (<\mathbf{u}> \times <\mathbf{B}> + \boldsymbol{\epsilon}) - \partial <\mathbf{B}> /\partial t = 0. \quad (1)$$

Here the magnetic diffusivity η of the fluid is assumed to be constant. $\boldsymbol{\epsilon}$ is the mean electromotive force caused by fluctuations,

$$\boldsymbol{\epsilon} = <\mathbf{u}' \times \mathbf{B}'>; \quad (2)$$

angular brackets always indicate averages. For a wide range of reasonable assumptions the quantity ϵ, when represented in Cartesian coordinates, has the form

$$\epsilon_i = a_{ij} < \mathbf{B} >_j + b_{ijk}\, \partial < \mathbf{B} >_j / \partial x_k. \tag{3}$$

The tensorial coefficients a_{ij} and b_{ijk} depend on $< \mathbf{u} >$ and $\mathbf{u}'$. Within the framework of kinematic theory, to which we restrict ourselves in this paper, they are independent of $< \mathbf{B} >$. The first term on the right-hand side describes the α-effect, the second term various effects like that of the turbulent magnetic diffusivity. In the simple case in which $< \mathbf{u} >$ is equal to zero and $\mathbf{u}'$ corresponds to isotropic turbulence, relation (3) reduces to

$$\epsilon = \alpha < \mathbf{B} > - \beta \operatorname{curl} < \mathbf{B} >, \tag{4}$$

with scalar coefficients α and β which are determined by $\mathbf{u}'$. The first term on the right-hand side then describes the ideal, that is isotropic, α-effect and the second term an effect which is completely covered by introducing a turbulent magnetic diffusivity.

In dynamo processes the helicities of both the motion and the magnetic field are of interest. In the following we pay particular attention to the *current helicity*, $\mathbf{B} \cdot \operatorname{curl} \mathbf{B}$, the mean value of which, $< \mathbf{B} \cdot \operatorname{curl} \mathbf{B} >$, can be represented as the sum

$$< \mathbf{B} \cdot \operatorname{curl} \mathbf{B} > = < \mathbf{B} > \cdot \operatorname{curl} < \mathbf{B} > + < \mathbf{B}' \cdot \operatorname{curl} \mathbf{B}' > \tag{5}$$

of two contributions resulting from the mean and fluctuating magnetic field. In this paper we show that there is, at least in some approximation, a simple connection between the current helicity $< \mathbf{B}' \cdot \operatorname{curl} \mathbf{B}' >$ of the fluctuating magnetic field and the α-effect coefficients a_{ij}, or α, which are in turn related to the kinematic helicity of the fluctuating motions. Using this result we study the current helicities $< \mathbf{B}' \cdot \operatorname{curl} \mathbf{B}' >$ and $< \mathbf{B} > \cdot \operatorname{curl} < \mathbf{B} >$ of the fluctuating and mean magnetic fields in α^2-dynamos. It is shown that the energy of the mean magnetic field can only be maintained or grow if there is a region in which the signs of $< \mathbf{B}' \cdot \operatorname{curl} \mathbf{B}' >$ and $< \mathbf{B} > \cdot \operatorname{curl} < \mathbf{B} >$ are different and, in addition, $| < \mathbf{B}' \cdot \operatorname{curl} \mathbf{B}' > |$ exceeds $| < \mathbf{B} > \cdot \operatorname{curl} < \mathbf{B} > |$. Furthermore, we present some results on the current helicity in active regions on the Sun which have been derived from observational data, and discuss these in the light of our findings concerning current helicity and the α-effect.

2. CURRENT HELICITY OF THE FLUCTUATING MAGNETIC FIELD AND α-EFFECT

In order to derive a relation between the current helicity $<\mathbf{B}' \cdot \operatorname{curl} \mathbf{B}'>$ of the fluctuating magnetic field and the α-effect coefficients a_{ij}, or α, we restrict ourselves to the case in which $<\mathbf{u}>$ is zero and $\mathbf{u}'$ so small that the second order correlation approximation applies. Then we have

$$\eta \triangle \mathbf{B}' - \partial \mathbf{B}'/\partial t = -\operatorname{curl}(\mathbf{u}' \times <\mathbf{B}>). \qquad (6)$$

For the determination of $\boldsymbol{\epsilon}$ and also of $<\mathbf{B}' \cdot \operatorname{curl} \mathbf{B}'>$ it is useful to subject $\mathbf{B}'$ and $\mathbf{u}'$ to a Fourier transformation of the form

$$F(\mathbf{x}, t) = \int\int \hat{F}(\mathbf{k}, \omega) e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} d^3\mathbf{k}\, d\omega. \qquad (7)$$

Then (6), with $<\mathbf{B}>$ taken as constant, reduces to

$$\hat{\mathbf{B}}'_i = i\, \epsilon_{ijk}\, \epsilon_{klm} \frac{k_j \hat{u}'_l <\mathbf{B}>_m}{\eta k^2 - i\omega}. \qquad (8)$$

We further assume that $\mathbf{u}'$ describes a homogeneous and steady field of turbulence. This implies that

$$<\hat{u}'_i(\mathbf{k}, \omega) \hat{u}'_j(\mathbf{k}', \omega')> = \hat{Q}_{ij}(\mathbf{k}', \omega')\, \delta(\mathbf{k} + \mathbf{k}')\, \delta(\omega + \omega'), \qquad (9)$$

where $\hat{Q}_{ij}(\mathbf{k}, \omega)$ is the Fourier transform of the correlation tensor $Q_{ij}(\boldsymbol{\xi}, \tau)$ defined by

$$Q_{ij}(\boldsymbol{\xi}, \tau) = <u'_i(\mathbf{x}, t)\, u'_j(\mathbf{x} + \boldsymbol{\xi},\ t + \tau)> \qquad (10)$$

A straightforward calculation of $\boldsymbol{\epsilon}$ using (8) and (9) provides us with

$$a_{ij} <\mathbf{B}>_i <\mathbf{B}>_j = -\eta \int\int \frac{(\mathbf{G}(\mathbf{k}, \omega)\cdot <\mathbf{B}>)(\mathbf{k}\cdot <\mathbf{B}>) k^2}{(\eta k^2)^2 + \omega^2} d^3k\, d\omega. \qquad (11)$$

In an analogous way we find

$$<\mathbf{B}' \cdot \operatorname{curl} \mathbf{B}'> = \int\int \frac{(\mathbf{G}(\mathbf{k}, \omega)\cdot <\mathbf{B}>)(\mathbf{k}\cdot <\mathbf{B}>) k^2}{(\eta k^2)^2 + \omega^2} d^3k\, d\omega. \qquad (12)$$

In both cases G is given by

$$G_i(\mathbf{k},\omega) = -i\ \epsilon_{ijk}\hat{Q}_{jk}(k,w). \tag{13}$$

$\mathbf{G}(\mathbf{k},w)\cdot\mathbf{k}$ is just the Fourier transform of the kinematic helicity spectrum function $H(\xi,\tau)$ defined by

$$H(\xi,\tau) = <\mathbf{u}'(\mathbf{x},t)\cdot\ \mathrm{curl}\ \mathbf{u}'(\mathbf{x}+\xi,\ t+\tau)>. \tag{14}$$

We note that (11) can also readily be derived from a relation mentioned by Krause and Rädler (1980, eq. 7.1), and (12) from a result by Bräuer and Krause (1972, eq. 14) or, for incompressible fluids, from a result by Rüdiger (1974, eq. 26).

Comparing (11) and (12) we see that

$$<\mathbf{B}'\cdot\ \mathrm{curl}\ \mathbf{B}'> = -(1/\eta)a_{ij}<\mathbf{B}>_i<\mathbf{B}>_j. \tag{15}$$

For the special case of isotropic turbulence, for which $a_{ij} = \alpha\delta_{ij}$, we have simply

$$<\mathbf{B}'\cdot\ \mathrm{curl}\ \mathbf{B}'> = -(\alpha/\eta)<\mathbf{B}>^2. \tag{16}$$

Relations of this type have already been given by Keinigs (1983) and by Mattheaus et al (1986).

As is well-known, the sign of α is, as a rule, opposite to that of the kinematic helicity of the fluctuating motions. Hence, as expected, the sign of the current helicity of the fluctuating magnetic field coincides with that of the kinematic helicity of the motions responsible for them.

3. CURRENT HELICITIES IN AN α^2-DYNAMO

Let us now consider a fluid body surrounded by free space and assume that the mean magnetic field inside this body is governed by (1) and continues as an irrotational field in the external space. Then we have

$$\frac{d}{dt}\int_\infty \frac{1}{2}<\mathbf{B}>^2 dV = -\int (\eta\ \mathrm{curl}^2<\mathbf{B}>+$$
$$+<\mathbf{u}>\cdot(\ \mathrm{curl}\ <\mathbf{B}>\times<\mathbf{B}>)+\ \mathrm{curl}\ <\mathbf{B}>\cdot\epsilon)\,dV, \tag{17}$$

where the integral on the left is over all space and that on the right over the fluid body. The energy stored in the mean magnetic field is maintained, or it grows, if the integral on the right is zero, or negative.

We restrict attention to α^2-dynamos and assume that ϵ is given by (4) with non-negative β. Thinking of a proper frame of reference we further put $<\mathbf{u}>=0$. We require that the energy of the mean magnetic field does not decay, that is,

$$\int \left(\alpha <\mathbf{B}> \cdot \text{curl } <\mathbf{B}> - (\eta+\beta) \text{ curl}^2 <\mathbf{B}>\right) dV \geq 0. \tag{18}$$

This implies that there must be sufficiently extended regions of the fluid in which the signs of α and $<\mathbf{B}> \cdot \text{curl } <\mathbf{B}>$ coincide. As long as (16) applies, in these regions the signs of the current helicities $<\mathbf{B}> \cdot \text{curl } <\mathbf{B}>$ and $<\mathbf{B}' \cdot \text{curl } \mathbf{B}'>$ of the mean and the fluctuating magnetic fields are different.

In order to deduce a further result concerning the current helicities we start again from (18) and introduce

$$<\mathbf{B}' \cdot \text{curl } \mathbf{B}'> = -f <\mathbf{B}> \cdot \text{curl } <\mathbf{B}> \tag{19}$$

with a factor f depending on the space coordinates. In this way we arrive at

$$\int f\left((<\mathbf{B}> \cdot \text{curl } <\mathbf{B}>)^2/<\mathbf{B}>^2\right) dV \geq \int (1+\beta/\eta)(\text{ curl } <\mathbf{B}>)^2 dV \tag{20}$$

According to a mean value theorem of integral calculus, f takes somewhere in the fluid volume a value f^* such that

$$\int f \frac{(<\mathbf{B}> \cdot \text{curl } <\mathbf{B}>)^2}{<\mathbf{B}>^2} dV = f^* \int \frac{(<\mathbf{B}> \cdot \text{curl } <\mathbf{B}>)^2}{<\mathbf{B}>^2} dV. \tag{21}$$

From (20) and (21) we have

$$f^* \geq \int (1+\beta/\eta)(\text{ curl } <\mathbf{B}>)^2 dV \Big/ \int \left((<\mathbf{B}> \cdot \text{curl } <\mathbf{B}>)^2/<\mathbf{B}>^2\right) dV \tag{22}$$

and therefore

$$f^* \geq 1. \tag{23}$$

Assuming continuity of all relevant quantities, we conclude that the energy of the mean magnetic field can only be prevented from decaying if there is some region in the fluid where the signs of $< \mathbf{B}' \cdot \text{curl } \mathbf{B}' >$ and $< \mathbf{B} > \cdot \text{curl } < \mathbf{B} >$ differ and, in addition, $| < \mathbf{B}' \cdot \text{curl } \mathbf{B}' > | > | < \mathbf{B} > \cdot \text{curl } < \mathbf{B} > |$.

4. CURRENT HELICITY IN SOLAR ACTIVE REGIONS

Let us now proceed to an application of our relations (15) and (16) connecting the current helicity of fluctuating magnetic fields and the α-effect to solar phenomena. The solar magnetic fields are attributed to an $\alpha\omega$-dynamo operating in the convection zone of the Sun, that is, below the visible surface. In the observable atmosphere, besides weak background fields, strong magnetic fields are found in active regions. These regions are believed to result from the emergence of magnetic flux ropes which have broken away from the predominantly toroidal field below the surface and carried up by magnetic buoyancy. Above the photosphere (the thin layer at the base of the atmosphere which represents the surface of the Sun in white light), the magnetic energy density in active regions dominates, except for explosive events (such as flares) over the thermal, kinetic and gravitational energy densities. Therefore a quasi-static equilibrium of the plasma may be assumed with a force-free magnetic field. Of course, there is some evolution of the configuration, which is induced by plasma motions in or below the photosphere. However, these motions are slow compared to the Alfven velocity in the superphotospheric plasma, that is, small compared to the velocity of the upward propagation of disturbances caused by them, so that each state may be considered as an equilibrium state (cf. Low, 1982). Since then the magnetic field above the photosphere should be force-free, we have there

$$\text{curl } \mathbf{B} = \alpha_{ff} \mathbf{B}, \tag{24}$$

for some pseudo-scalar α_{ff}. The sign of the current helicity $\mathbf{B} \cdot \text{curl } \mathbf{B}$ coincides with that of α_{ff}.

Seehafer (1989) has compiled 16 active regions for which the factor α_{ff} was estimated. Using observed photospheric magnetic fields as boundary data force-free fields with constant α_{ff} in the volume above the photosphere were calculated and α_{ff} was varied until an optimum coincidence of the calculated field line configrations with observed superphotospheric structures believed to be aligned with the field was obtained. Of the 16 regions, which belonged to the activity cycles 20 (beginning 1965) and 21 (beginning 1976), 12 lay in the northern and 4 in the southern hemisphere. For 11 of the 12 regions in the northern hemisphere α_{ff} was negative, for one positive. In the southern hemisphere, in 3 cases α_{ff} was positive, in one case a change of the sign of α_{ff} within the region was suggested. Thus we are led to conclude that in solar active regions the current helicity is predominantly negative in the northern and positive in the southern hemisphere (though further investigations are needed to firmly establish this result).

In the traditional mean-field concept of the solar dynamo the mean magnetic field does not reflect the magnetic fields of the individual active regions. Although these fields may contribute to the mean field, they are presumably mainly fluctuating fields. Their helicities then have to be interpreted in the sense of $< \mathbf{B}' \cdot \text{curl}\, \mathbf{B}' >$. Using the relation (15) we thus conclude that $a_{ij} < \mathbf{B} >_i < \mathbf{B} >_j$ is predominantly positive in the northern and negative in the southern hemisphere. If the α-effect is assumed to be isotropic we may replace (15) by (16) and conclude that α is mainly positive in the northern and negative in the southern hemisphere. This corresponds to the usual picture used in many solar dynamo models (see, e.g., Krause and Rädler , 1980). Of course, the α-effect in the Sun deviates from isotropy. Taking this into account and assuming that the toroidal component of the mean magnetic field is large compared to the other components, we may conclude that $\alpha_{ij} < \mathbf{B} >_i < \mathbf{B} >_j$ is approximately equal to $\alpha_{\varphi\varphi} < \mathbf{B} >^2$ where $\alpha_{\varphi\varphi}$ is just that component of the α-tensor which is responsible for the regeneration of the poloidal from the toroidal field. Clearly, the sign rule formulated above for the scalar α then applies to $\alpha_{\varphi\varphi}$, too.

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