CORONAL FORCE-FREE MAGNETIC FIELD: SOURCE-SURFACE MODEL

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Abstract. Models of the magnetic field in the solar chromosphere and corona are still mainly based on theoretical extrapolations of photospheric measurements. For the practical calculation of the global field, the so-called source-surface model has been introduced, in which the influence of the solar wind is described by the requirement that the field be radial at some exterior (source) surface. Then the assumption that the field is current-free in the volume between the photosphere and this surface allows for its determination from the photospheric measurement. In the present paper a generalization of the source-surface model to force-free fields is proposed. In the generalized model the parameter $\alpha\,(=\nabla \times \mathbf{B}\cdot\mathbf{B}/\mathbf{B}^2)$ *must* be non-constant (or vanish identically) and currents are restricted to regions with closed field lines. A mathematical algorithm for computing the field from boundary data is devised.

1. Introduction

Reliable measurements of solar magnetic fields still being restricted to the photosphere, models of the magnetic field in the chromosphere and in the corona are mainly based on theoretical extrapolations of photospheric measurements. When, instead of the near-surface configuration above-limited photospheric regions, the global field configuration is considered, the interaction of the solar wind with the solar magnetic field must be taken into account. This interaction has been treated selfconsistently only for the case of a photospheric dipole field (Pneuman and Kopp, 1971; Weber, 1978; Steinolfson, Suess, and Wu, 1982). For modelling more complex fields, Schatten, Wilcox, and Ness (1969) and Altschuler and Newkirk (1969) introduced the so-called source-surface model, in which the plasma-magnetic field coupling is expressed by the requirement that the field be radial at some spherical surface $r = R_S$ (between 1.5 $R_\odot$ and 2.5 $R_\odot$). The magnetic field in the volume between the photosphere and this surface is assumed to be current-free, thus allowing for its determination from the measured photospheric line-of-sight component.

Some mathematical aspects of this model have been considered by Aly (1987). Concerning the computational technique, improvements have been proposed by Adams and Pneuman (1976), Altschuler *et al.* (1977), and Riesebieter and Neubauer (1979). Also non-spherical source-surfaces are used (Schulz, Frazier, and Boucher, 1978;

Levine, Schulz, and Frazier, 1982). Yeh and Pneuman (1977) modelled, starting from a current-free field, the effect of current sheets between closed and open field lines and between oppositely directed open field lines on the magnetic field.

Volume currents, on the other hand, must be force-free, at least in the inner part of the atmosphere, where the magnetic pressure dominates over both the gas pressure and the kinetic energy density of the plasma flow. That is, electric currents must be aligned with the magnetic field and we have

$$\nabla \times \mathbf{B} = \alpha \mathbf{B} ,$$

with α denoting a pseudo-scalar function.

Models of global force-free fields have been restricted to spatially constant α. Nakagawa (1973) and Nakagawa, Wu, and Tandberg-Hanssen (1978) assumed the magnetic field to be force-free with spatially constant α in the whole exterior of the photospheric surface. This, however, is an inappropriate assumption since a magnetic field being force-free with constant α ($\neq 0$) in the whole exterior of a bounded volume cannot have a finite energy content (Seehafer, 1978).

Elwert *et al.* (1982) put tangential planes at centres of strong magnetic fields, computed the constant-α force-free field above each center by using a Green's function solution for the half space given by Chiu and Hilton (1977) (for the current-free field, $\alpha = 0$, the solution of Schmidt (1964) was used) and added these individual fields. This model suffers from bad discontinuities of the total field across the boundary planes of the half spaces in those regions where different half spaces overlap (for the individual fields of overlapping half spaces the value of α must be the same since otherwise the total field is not force-free). Barbosa (1978) proposed to prescribe in addition to the photospheric boundary condition the normal field component on an outer spherical surface and to calculate the constant-α force-free field in the volume between the two surfaces. But the required information on the normal field component at the outer surface is presently not available.

Durrant (1989), who tried to generalize the source-surface model to constant-α force-free fields, noticed that the condition of a purely radial field at $r = R_S$ plus the further condition on the photosphere overdetermine the problem. He proposed to minimize the horizontal field on the outer boundary in the least-squares sense, subject to the constraint provided by the inner boundary condition. Indeed a constant-α force-free field which is perpendicular to some spherical surface must be a potential field ($\alpha = 0$). This is part of a more general statement derived in Section 2.3.

In the present paper we propose a generalization of the source-surface model to non-constant-α ('nonlinear') force-free fields. We also devise a mathematical method for computing the field from boundary data. In this point we adapt previous work of Bineau (1972) and Sakurai (1981).

2. Statement of the Problem

2.1. Assumptions

(i) Let the corona be defined as the fixed domain D bounded by a sphere S_1 of radius R_1 and an outer simple closed regular surface S_2 and let $\hat{\mathbf{n}}$ denote the interior unit normal on the boundary $\partial D = S_1 \cup S_2$ of D (on S_1 then $\hat{\mathbf{n}} = \hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is the unit vector in the radial direction with respect to an origin in the center of S_1). On ∂D, a vector $\mathbf{x}$ is decomposed into its normal and tangential components $\mathbf{x}_n = x_n \hat{\mathbf{n}}$ and $\mathbf{x}_t$:

$$\mathbf{x}_n = (\mathbf{x} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} \quad \text{on } \partial D , \tag{1}$$

$$\mathbf{x}_t = \mathbf{x} - x_n \hat{\mathbf{n}} \quad \text{on } \partial D . \tag{2}$$

(ii) The magnetic field $\mathbf{B}$ is assumed to be force-free in D, i.e., to satisfy

$$\nabla \times \mathbf{B} = \alpha \mathbf{B} \quad \text{in } D , \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \text{in } D . \tag{4}$$

Equations (3) and (4) imply $\mathbf{B} \cdot \nabla \alpha = 0$ in D, i.e., α is constant along the field lines or the field lines lie in the surfaces $\alpha =$ constant, respectively.

(iii) The normal component $B_r = B_n$ of $\mathbf{B}$ on S_1 is equal to a given function q_1:

$$B_n = q_1 \quad \text{on } S_1 , \tag{5}$$

$$\int_{S_1} q_1 \, d\sigma = 0 . \tag{6}$$

Here and in the following, $d\sigma$ denotes a surface element.

(iv) The tangential component $\mathbf{B}_t$ of $\mathbf{B}$ on S_2 is assumed to vanish:

$$\mathbf{B}_t = 0 \quad \text{on } S_2 , \tag{7}$$

i.e., $\mathbf{B} \perp S_2$ on S_2.

2.2. Field lines of B

We define as $\mathscr{C}(\mathbf{r})$ the field line of $\mathbf{B}$ passing through the point $\mathbf{r}$. The function q_2 denotes the values of B_n on S_2, which, in contrast to the values q_1 of B_n on S_1, are not prescribed but depend on $\mathbf{B}$:

$$q_2 = B_n \quad \text{on } S_2 . \tag{8}$$

Furthermore, we define the following subsets of S_1 and S_2 (Figure 1):

$$S_i^+ = \{\mathbf{r} \mid \mathbf{r} \in S_i, q_i(\mathbf{r}) > 0\} , \quad i = 1, 2 , \tag{9}$$

$$S_i^- = \{\mathbf{r} \mid \mathbf{r} \in S_i, q_i(\mathbf{r}) < 0\} , \quad i = 1, 2 , \tag{10}$$

$$S_{ic}^\pm = \{\mathbf{r} \mid \mathbf{r} \in S_i^\pm , \mathscr{C}(\mathbf{r}) \text{ cuts } S_i^\mp\} , \quad i = 1, 2 , \tag{11}$$

$$S_{i\,op}^\pm = \{\mathbf{r} \mid \mathbf{r} \in S_i^\pm , \mathscr{C}(\mathbf{r}) \text{ cuts } S_{j(i)}^\mp,\} , \quad i = 1, 2, \ j(1) = 2, \ j(2) = 1 . \tag{12}$$

Like q_2, also $S_2^{\pm}$, $S_{ic}^{\pm}$ and $S_{i\,op}^{\pm}$ depend on **B**.

Finally, two subvolumes of D are defined, which depend on **B** as well:

$$D_{ic} = \{\mathbf{r} \mid \mathbf{r} \in D,\ \mathscr{C}(r) \text{ cuts } S_i^+ \text{ and } S_i^-\}, \quad i = 1, 2 . \tag{13}$$

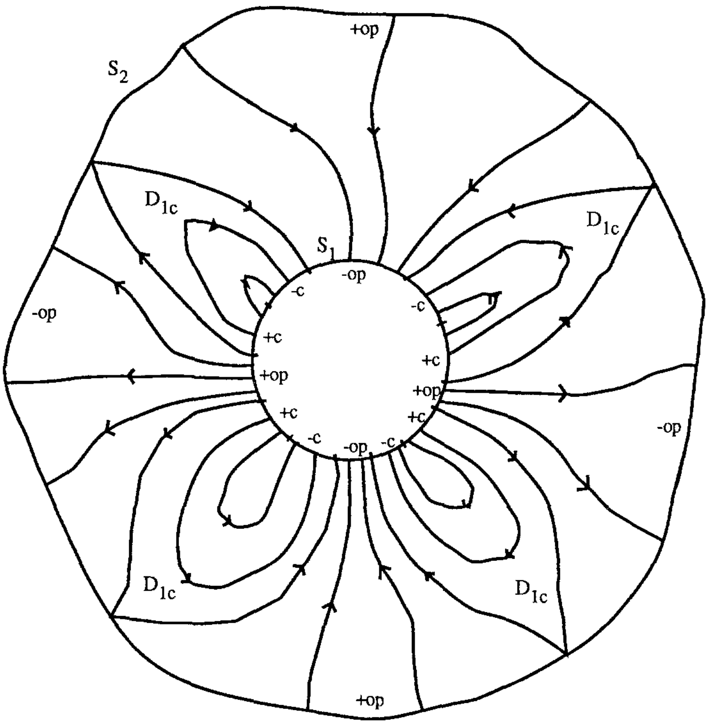

Fig. 1. Schematic view of a possible configuration in our model. The magnetic corona is represented by the space between the two surfaces S_1 (photosphere) and S_2 (source surface). It is divided into two parts: one containing lines connecting S_1 and S_2 (or, more precisely, connecting either S_{1op}^+ to S_{2op}^- or S_{1op}^- to S_{2op}^+) and one (D_{1c}) containing lines connecting the two parts S_{1c}^+ and S_{1c}^- of S_1. Lines with arrows symbolize magnetic field lines.

2.3. A basic property of B

For any field **B** on any surface it is

$$\nabla \times \mathbf{B} = (\nabla_t + \nabla_n) \times (\mathbf{B}_t + \mathbf{B}_n) = \nabla_t \times \mathbf{B}_t + \nabla_t \times \mathbf{B}_n + \nabla_n \times \mathbf{B}_t , \tag{14}$$

since

$$\nabla_n \times \mathbf{B}_n = \hat{\mathbf{n}} \frac{\partial}{\partial n} \times B_n \hat{\mathbf{n}} = \hat{\mathbf{n}} \times \frac{\partial \mathbf{B}_n}{\partial n} \hat{\mathbf{n}} = 0 .$$

For the normal component of $\nabla \times \mathbf{B}$ we then have

$$\hat{\mathbf{n}} \cdot (\nabla \times \mathbf{B}) = \hat{\mathbf{n}} \cdot (\nabla_t \times \mathbf{B}_t) , \tag{15}$$

since

$$\hat{\mathbf{n}} \cdot (\nabla_t \times \mathbf{B}_n) = \hat{\mathbf{n}} \cdot (\nabla_t \times B_n \hat{\mathbf{n}}) = \hat{\mathbf{n}} \cdot (\nabla_t B_n \times \hat{\mathbf{n}}) = 0$$

and

$$\hat{\mathbf{n}} \cdot (\nabla_n \times \mathbf{B}_t) = \hat{\mathbf{n}} \cdot \left(\hat{\mathbf{n}} \frac{\partial}{\partial n} \times \mathbf{B}_t \right) = \hat{\mathbf{n}} \cdot \left(\hat{\mathbf{n}} \times \frac{\partial}{\partial n} \mathbf{B}_t \right) = 0 .$$

Because of the assumption (7), Equation (15) implies

$$\hat{\mathbf{n}} \cdot (\nabla \times \mathbf{B}) = 0 \quad \text{on } S_2 . \tag{16}$$

Consequently, according to Equation (3),

$$\alpha B_n = 0 \quad \text{on } S_2 . \tag{17}$$

On S_2, B_n can vanish only on a set of measure zero. Otherwise, we would have $\mathbf{B} = 0$ on a finite part of S_2. If $\mathbf{B}$ vanishes on some surface, however, then the three-dimensional force-free extension of the field from this surface must identically vanish (cf. Aly, 1989). Therefore,

$$\alpha = 0 \quad \text{on } S_2 \tag{18}$$

and, because of $\mathbf{B} \cdot \nabla\alpha = 0$,

$$\alpha = 0 \quad \text{on } S_{1op}^{\pm} . \tag{19}$$

2.4. Boundary value problem for B

Let h be a given function defined on S_1^+. We impose on $\mathbf{B}$ the supplementary condition (compatible with the result of Section 2.3)

$$\alpha = h \quad \text{on } S_{1c}^{+}[\mathbf{B}] ,$$

$$\alpha = 0 \quad \text{in } D \backslash D_{1c}[B] .$$

Then $\mathbf{B}$ must be a solution of the following boundary value problem (BVP):

$$\nabla \times \mathbf{B} = \alpha \mathbf{B} \quad \text{in } D , \tag{20a}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \text{in } D , \tag{20b}$$

$$B_n = q_1 \quad \text{on } S_1 , \tag{20c}$$

$$\mathbf{B}_t = 0 \quad \text{on } S_2 , \tag{20d}$$

$$\alpha = h \quad \text{on } S_{1c}^{+}[\mathbf{B}] , \tag{20e}$$

$$\alpha = 0 \quad \text{in } D \backslash D_{1c}[\mathbf{B}] . \tag{20f}$$

3. Associated Linear Problems

3.1. Potential field

(i) The potential field $\mathbf{B}_0$ is defined as the solution of the BVP:

$$\nabla \times \mathbf{B}_0 = 0 \quad \text{in } D , \tag{21a}$$

$$\nabla \cdot \mathbf{B}_0 = 0 \quad \text{in } D, \tag{21b}$$

$$B_{0n} = q_1 \quad \text{on } S_1, \tag{21c}$$

$$\mathbf{B}_{0t} = 0 \quad \text{on } S_2. \tag{21d}$$

(ii) We can write

$$\mathbf{B}_0 = \nabla \phi_0, \tag{22}$$

where ϕ_0 is a solution of the BVP:

$$\Delta \phi_0 = 0 \quad \text{in } D, \tag{23a}$$

$$\frac{\partial \phi_0}{\partial n} = q_1 \quad \text{on } S_1, \tag{23b}$$

$$\phi_0 = 0 \quad \text{on } S_2. \tag{23c}$$

The solution of this *mixed* BVP is well known to exist and to be unique.

(iii) The energy of $\mathbf{B}_0$ is the minimum value of the energies of all fields $\mathbf{B}$ satisfying $B_n = q_1$ on S_1. (Note that no condition is imposed on $\mathbf{B}$ on S_2!)

Proof:

We write

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{b} = \nabla \phi_0 + \mathbf{b} \quad \text{in } D, \tag{24a}$$

$$\nabla \cdot \mathbf{b} = 0 \quad \text{in } D, \tag{24b}$$

$$b_n = 0 \quad \text{on } S_1. \tag{24c}$$

For the magnetic energy we have

$$\int_D \mathbf{B}^2 \, \mathrm{d}V = \int_D (\mathbf{B}_0^2 + \mathbf{b}^2) \, \mathrm{d}V + 2 \int_D \mathbf{B}_0 \cdot \mathbf{b} \, \mathrm{d}V. \tag{25}$$

By making use of the divergence theorem and of the conditions (24b), (24c), and (23c) we obtain for the second term on the right-hand side of Equation (25)

$$\int_D \mathbf{B}_0 \cdot \mathbf{b} \, \mathrm{d}V = \int_D \nabla \phi_0 \cdot \mathbf{b} \, \mathrm{d}V = \int_D \nabla \cdot (\phi_0 \mathbf{b}) \, \mathrm{d}V = - \int_{\partial D} \phi_0 b_n \, \mathrm{d}V =$$

$$= - \int_{S_1} \phi_0 b_n \, \mathrm{d}\sigma - \int_{S_2} \phi_0 b_n \, \mathrm{d}\sigma = 0. \tag{26}$$

Thus

$$\int_D \mathbf{B}^2 \, \mathrm{d}V = \int_D (\mathbf{B}_0^2 + \mathbf{b}^2) \, \mathrm{d}V \geq \int_D B_0^2 \, \mathrm{d}V, \quad \text{q.e.d.} \tag{27}$$

(iv) The sets $S_{2c}^{\pm}[\mathbf{B}_0]$ are empty.

Proof:

Assume that S_{2c}^{+} and S_{2c}^{-} are not empty and consider a field line $\mathscr{C}_1$ of $\mathbf{B}_0$ connecting S_{2c}^{+} to S_{2c}^{-} (then $|\mathbf{B}_0| > 0$ on $\mathscr{C}_1$). Let $\mathscr{L}_1$ denote a line on S_2 connecting the endpoints of $\mathscr{C}_1$, so that $\mathscr{C}_1 + \mathscr{L}_1$ is closed. According to Stokes' theorem and condition (21a)

$$\int_{\mathscr{C}_1} \mathbf{B}_0 \cdot \mathrm{d}\mathbf{s} + \int_{\mathscr{L}_1} \mathbf{B}_{0t} \cdot \mathrm{d}\mathbf{s} = 0\,, \tag{28}$$

where $\mathrm{d}\mathbf{s}$ is a line element. Because of condition (21d) then

$$\int_{\mathscr{C}_1} \mathbf{B}_0 \cdot \mathrm{d}\mathbf{s} = 0\,, \tag{29}$$

$$\mathbf{B}_0 \equiv 0 \quad \text{on} \quad \mathscr{C}_1\,, \tag{30}$$

in contradiction to the assumption made, q.e.d.

(v) The energy of $\mathbf{B}_0$ in D may be expressed by

$$\int_D \mathbf{B}_0^2 \,\mathrm{d}V = \int_D (\nabla\phi_0)^2 \,\mathrm{d}V = -\int_D \phi_0 \Delta\phi_0 \,\mathrm{d}V + \frac{1}{2}\int_D \Delta\phi_0^2 \,\mathrm{d}V =$$
$$= \frac{1}{2}\int_D \Delta\phi_0^2 \,\mathrm{d}V = -\int_{\partial D} \phi_0 \frac{\partial\phi_0}{\partial n}\,\mathrm{d}\sigma = -\int_{S_1} \phi_0 q_1 \,\mathrm{d}\sigma\,. \tag{31}$$

3.2. Field generated by given currents

(i) The electric current density $\mathbf{j}$ is assumed to be prescribed in D, with

$$\nabla \cdot \mathbf{j} = 0 \quad \text{in } D \tag{32a}$$

and

$$j_n = 0 \quad \text{on } S_2 \tag{32b}$$

according to Equation (18).

We wish to determine $\mathbf{B}$ as solution of the BVP:

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\,\mathbf{j} \quad \text{in } D\,, \tag{33a}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{in } D\,, \tag{33b}$$

$$B_n = q_1 \qquad \text{on } S_1\,, \tag{33c}$$

$$B_t = 0 \qquad \text{on } S_2\,. \tag{33d}$$

This linear BVP has at most one solution since any solution of the associated homogeneous problem has to satisfy the same equations as $\mathbf{B}_0$, namely Equations (21a–d), but with q_1 replaced by 0 on the right-hand side of Equation (21c), and therefore must vanish identically.

(ii) To construct a solution $\mathbf{B}$ we first arbitrarily fix the values q_2 of B_n on S_2 and solve the BVP:

$$\nabla \times \mathbf{B}_1 = \frac{4\pi}{c}\,\mathbf{j} \quad \text{in } D\,, \tag{34a}$$

$$\nabla \cdot \mathbf{B}_1 = 0 \qquad \text{in } D\,, \tag{34b}$$

$$B_{1n} = q_1 \qquad \text{on } S_1\,, \tag{34c}$$

$$B_{1n} = q_2 \qquad \text{on } S_2\,. \tag{34d}$$

This BVP is well known to admit a unique solution. Because of Equation (32b),

$$\hat{\mathbf{n}} \cdot \nabla \times \mathbf{B}_1 = \frac{4\pi}{c}\, j_n = 0 \quad \text{on } S_2\,. \tag{35}$$

Therefore,

$$\mathbf{B}_{1t} = \nabla_t \phi_1 \quad \text{on } S_2 \tag{36}$$

with some function ϕ_1 defined on S_2.

We next consider the BVP:

$$\Delta\phi_2 = 0 \qquad \text{in } D\,, \tag{37a}$$

$$\frac{\partial \phi_2}{\partial n} = 0 \qquad \text{on } S_1\,, \tag{37b}$$

$$\phi_2 = -\phi_1 \quad \text{on } S_2\,, \tag{37c}$$

which admits a unique solution ϕ_2.

Now we set

$$\mathbf{B} = \mathbf{B}_1 + \nabla\phi_2\,. \tag{38}$$

Then

$$\nabla \times \mathbf{B} = \nabla \times \mathbf{B}_1 = \frac{4\pi}{c}\,\mathbf{j} \qquad \text{in } D\,, \tag{39a}$$

$$\nabla \cdot \mathbf{B} = \nabla \cdot \mathbf{B}_1 + \Delta\phi_2 = 0 \qquad \text{in } D\,, \tag{39b}$$

$$B_n = B_{1n} + \frac{\partial \phi_2}{\partial n} = q_1 \qquad \text{on } S_1\,, \tag{39c}$$

$$\mathbf{B}_t = \mathbf{B}_{1t} + \nabla_t\phi_2 = \nabla_t(\phi_1 + \phi_2) = 0 \quad \text{on } S_2\,. \tag{39d}$$

Thus the field $\mathbf{B}$ defined by Equation (38) is the uniquely determined solution of our original BVP, Equations (33a–d).

4. Iterative Calculation of the Force-Free Field in D

4.1. Iterative scheme

We define the following iteration for $\mathbf{B}$, assuming $\mathbf{B}^{(n)}$ to be known. First we determine $\alpha^{(n)}$ in D by solving

$$\mathbf{B}^{(n)} \cdot \nabla\alpha^{(n)} = 0 \quad \text{in } D_{1c}[\mathbf{B}^{(n)}]\,, \tag{40a}$$

$$\alpha^{(n)} = h \quad \text{on } S_{1c}^{+}[\mathbf{B}^{(n)}]\,, \tag{40b}$$

$$\alpha^{(n)} = 0 \quad \text{in } D\backslash D_{1c}[\mathbf{B}^{(n)}]\,. \tag{40c}$$

Now we are able to calculate $\mathbf{B}^{(n+1)}$ by solving

$$\nabla \times \mathbf{B}^{(n+1)} = \alpha^{(n)}\mathbf{B}^{(n)} \quad \text{in } D\,, \tag{41a}$$

$$\nabla \cdot \mathbf{B}^{(n+1)} = 0 \quad \text{in } D\,, \tag{41b}$$

$$B_n^{(n+1)} = q_1 \quad \text{on } S_1\,, \tag{41c}$$

$$\mathbf{B}_t^{(n+1)} = 0 \quad \text{on } S_2\,. \tag{41d}$$

This BVP is formally equivalent to that defined by Equations (33a–d) and so admits a unique solution $\mathbf{B}^{(n+1)}$. Equation (40c) ensures that $\alpha^{(n)} = 0$ on S_2.

4.2. Convergence of the iteration

We start the iteration from the potential field,

$$\mathbf{B}^{(0)} = \mathbf{B}_0\,. \tag{42}$$

Our iterative scheme is a modification of a scheme due to Bineau (1972), who considered the field in a bounded domain D, as do we (whereas Sakurai (1981) calculated it in a half space), with B_n prescribed everywhere on ∂D and α prescribed on a part of ∂D. He was able to prove convergence under the following restricting assumptions:

(1) The values of α are prescribed on a *connected* part S_α of ∂D on which $B_n \geq b > 0$, and $\alpha \neq 0$ only on a connected subset S'_α strictly included in S_α.

(2) $|\mathbf{B}_0(\mathbf{r})| \geq b_0 > 0$ in all points of D connected to S_α by a field line of $\mathbf{B}_0$ (from which the iteration starts).

(3) $|\alpha|$ must be sufficiently small. More precisely, if we write

$$\alpha(\mathbf{r}) = \lambda\hat{\alpha}(\mathbf{r}) \tag{43}$$

with a spatially constant parameter λ, then convergence of the iteration sequence (and existence of a solution) is proven only in the neighborhood of $\lambda = 0$.

That is, electric currents are restricted to one simple magnetic flux tube that does not contain singular (null, neutral) points and on the end faces of which the normal field component does not change sign, and for small enough $|\lambda|$ this is guaranteed for all $\mathbf{B}^{(n)}$ and for the limit field of the iteration sequence. (Bineau explicitly requires the field lines of $\mathbf{B}_0$ starting from S_α to have their extremities on ∂D, i.e., to be of finite length in D.

We ensure this by prescribing the values of α only on S_{1c}^{+}.)

As to these restrictions and our modification of the scheme we make the following remarks:

(i) The proof of convergence remains valid if currents are allowed in a finite number of well separated simple flux tubes, viz., if the values of α are prescribed on a finite number of disjunct subsets $S_{\alpha}^{(k)}$, $k = 1, \ldots, K$, of ∂D (in our case of S_{1c}^{+}) with the above properties (1) and (2).

(ii) Instead of prescribing B_n on both S_1 and S_2 we prescribe B_n on S_1 and require $\mathbf{B}_t = \mathbf{0}$ on S_2. This does not influence Bineau's proof of convergence provided field lines along which currents flow are kept away from S_2. If we prescribe $\alpha \neq 0$ ($h \neq 0$) only on some disjunct subsets $S_{\alpha}^{(k)}$ *strictly included* in $S_{1c}^{+}[\mathbf{B}_0]$, then for sufficiently small $|\lambda|$ the $S_{\alpha}^{(k)}$ will stay inside $S_{1c}^{+}[\mathbf{B}^{(n)}]$ during the iteration and we are still in Bineau's context.

(iii) Without the requirements (1) and (2), difficulties may arise at separatrix surfaces across which the mapping from ∂D to ∂D defined by the field lines is discontinuous. Such discontinuities may be due to the presence of magnetic null points in D or of field lines tangential to ∂D (cf. Seehafer, 1986). Two field lines on opposite sides of a separatrix may have arbitrarily close endpoints on ∂D though they start from points on ∂D separated by a finite distance. If then different values of α are prescribed at the starting points, this must lead to a discontinuity of α across the separatrix. Even if the boundary values of α are prescribed in such a way that their continuation along the field lines of the potential field $\mathbf{B}_0$ does *not* lead to discontinuities, these will be observed already after the first iteration step, since the correspondence between points on ∂D defined by the field lines and the location of the separatrices change during the iteration, in contrast to the prescribed boundary values of α.

(iv) If such discontinuities of α and consequently of $\mathbf{j}$ appear, the iteration sequence can nevertheless be built: Let α_n (and $\mathbf{j}^{(n)} = (\alpha^{(n)}/(4\pi/c))\mathbf{B}^{(n)}$) be bounded and piecewise continuous in D. Then $\mathbf{B}$ can be calculated by solving Equations (41a–d) and will be continuous and even differentiable (but not continuously differentiable). It seems likely that also in this case the iteration sequence will converge, of course not to a *classical* but to a *generalized*, or *weak*, solution, viz., one with a finite number of surfaces across which the current density is discontinuous.

5. Discussion

(i) Under the approximation of a force-free magnetic field in the atmosphere, as shown in Section 2.3, the field must be current-free at the source surface. This implies that all open field lines originate from current-free regions in the lower atmosphere. Currents can then still flow along (1) closed field lines (field lines with two photospheric endpoints), (2) field lines leading from the photosphere to a singular point of the field and (3) field lines detached from both the photosphere and the source surface. In our model only currents along field lines of type (1) are allowed whereas field lines of types (2)–(3), if they should be present, are current-free.

(ii) The boundary surfaces between closed and open regions, e.g., in streamers, may or may not touch the source surface. In the second case the field lines forming these surfaces must each end in a magnetic null point situated in the interior of the volume between photosphere and source surface.

(iii) Since the solar wind escapes along open field lines, its source region must be current-free. One should note however that this is strictly valid only if the field is strictly force-free throughout the atmosphere.

(iv) Note also in this context that field lines terminating on the source surface do not necessarily start from the photosphere, that is, do not necessarily belong to one of the sets $S^{\pm}_{2op}$ defined in Section 2.2. They can also connect the source surface with a singular point of the field or have two endpoints on the source surface, though the latter kind of field lines is excluded for the potential field (cf. Section 3.1.).

(v) The finding that $\mathbf{j} = 0$ on a surface to which a force-free $\mathbf{B}$ is perpendicular cannot be generalized to non-force-free magnetohydrostatic equilibria, characterized by the equation

$$\mathbf{j} \times \mathbf{B} = \nabla p\,, \tag{44}$$

where p is the gas pressure. A simple counter-example is the θ-pinch (cf., e.g., Shercliff, 1965, p. 75). In this cylindrical system the current flow is azimuthal and produces an axial magnetic field. $\mathbf{B}$ is perpendicular to the cross-sectional surfaces of the cylinder, though these are not current-free.

(vi) We suppose that the iterative procedure for the calculation of the field does work even if currents are allowed to flow along field lines that are close to the separatrix surfaces across which the mapping from the boundary $\partial D = S_1 \cup S_2$ to itself defined by the field lines is discontinuous. Then the current density may become discontinuous across these surfaces. Such surfaces, e.g., separate regions with closed field lines (which have two photospheric endpoints) from regions with open field lines connecting the photosphere with the source surface. Since the location of the separatrices changes during the iteration, jumps of α from finite values in a closed-field region to zero in an open-field region will hardly be avoidable (unless the prescribed boundary values of α are changed in the course of the iteration).

(vii) Finally a remark about the specification of the photospheric boundary data seems in order. In principle, both B_n and α on S_1 can be obtained from vector magnetograms; α can be calculated according to

$$\alpha = \hat{\mathbf{n}} \cdot (\nabla_t \times \mathbf{B}_t)/B_n\,. \tag{45}$$

For the determination of the tangential field from the measurements the 180° ambiguity must be resolved somehow (see discussion of this problem by Aly, 1989). Also the numerical differentiation required in Equation (45) seems problematic since the accuracy of the horizontal field measurements is still rather low. In this context it may be helpful that the method proposed does not fully exhaust the information content of vector magnetograms so that there are some 'redundances' in the data. In contrast to α, B_n has to be specified on the whole photosphere. So if measurements are available only for a limited photospheric region and one is interested only in the local field above this region, nevertheless the entire global field must be calculated and, in particular, some assumption be made on B_n on the rest of the photosphere, in such a way that the total magnetic flux through the photosphere vanishes. (Note, however, that any method to calculate local fields from local measurements requires additional assumptions.)

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