# Computational Astrophysics I: Introduction and basic concepts

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# Applications: The Lane-Emden equation

# The Lane-Emden Equation I

We remember: Stellar structure equations

## Example: Boundary values

First two equations of stellar structure (e.g., for white dwarf), with mass coordinate $m$ (Lagrangian description)

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho} \quad \text{mass continuity, cf. shell } dm = 4\pi r^2 \rho dr \tag{1}$$

$$\frac{\partial P}{\partial m} = -\frac{GM}{4\pi r^4} \quad \text{hydrostatic equilibrium} \tag{2}$$

+ equation of state $P(\rho)$ (e.g., ideal gas $P(\rho, T) = RT\rho/\mu$), and boundary values

$$\text{center} \quad m = 0 : r = 0 \tag{3}$$

$$\text{surface} \quad m = M : \rho = 0 \rightarrow P = 0 \tag{4}$$

$\rightarrow$ solve for $r(m)$, specifically for $R_* = r(m = M_*)$, i.e. for given $M_*$

## Derivation of the Lane-Emden equation

(see also Hansen & Kawaler 1994)
$\rightarrow$ if equation of state (EOS) for pressure is only function of density, e.g., completely degenerate, nonrelativistic, electron gas (e.g., white dwarf)

$$P_\mathrm{e} = 1.004 \times 10^{13} \left(\frac{\rho[\mathrm{g\,cm}^{-3}]}{\mu_\mathrm{e}}\right)^{5/3} \mathrm{dyn\,cm}^{-2} \tag{5}$$

so, $P \propto (\rho/\mu_\mathrm{e})^{5/3}$ power law ...

($\mu_\mathrm{e} = [\sum Z_i\, X_i\, y_i/A_i]^{-1}$ mean molecular weight per electron, e.g., $\mu_\mathrm{e} \approx (\frac{1\cdot 0.7\cdot 1}{1} + \frac{2\cdot 0.3\cdot 1}{4}) \approx 1.2$ for fully ionized H-He plasma)

Polytropes are pseudo-stellar models where a power law for $P(\rho)$ is assumed a priori without reference to heat transfer/thermal balance

$\rightarrow$ only hydrostatic and mass continuity equation taken into account define a polytrope as

$$P(r) = K\rho^{1+\frac{1}{n}}(r) \tag{6}$$

with some constant $K$ and the *polytropic index* $n$.
→ polytrope must be in hydrostatic equlibrium, so hydrostatic equation (function of $r$ only)

$$\frac{dP}{dr} = -\frac{GM_r}{r^2}\rho \quad \Big| \cdot \frac{r^2}{\rho} \;\Big|\; d/dr \tag{7}$$

with the continuity equation $\frac{dM_r}{dr} = 4\pi r^2\rho$ and the mass variable $M_r = \int_0^r dm(r)$, i.e., $M_r = 0$ → center ($r = 0$, $\rho = \rho_\mathrm{c}$) and $M_r = M_*$ → surface ($r = R_*$, $\rho = 0$)

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\frac{dM_r}{dr} = -4\pi G r^2\rho \tag{8}$$

so finally:

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho \tag{9}$$

$\rightarrow$ Poisson's equation of gravitation with $g(r) = d\Phi/dr = GM_r/r^2$, and $\frac{dP}{dr} = -\frac{GM_r}{r^2}\rho$
hence $\rightarrow$ $\nabla^2\Phi = 4\pi G\rho$ in spherical coordinates

find transformations to make Eq. (9) *dimensionless*. Define *dimensionless* variable $\theta$ by

$$\rho(r) = \rho_\mathrm{c}\theta^n(r) \tag{10}$$

$\rightarrow$ then, power law for pressure from our definition of the polytrope Eq. (6)

$$P(r) = K\rho^{1+1/n}(r) = K\rho_\mathrm{c}^{1+1/n}\theta^{n+1}(r) = P_\mathrm{c}\theta^{1+n}(r) \tag{11}$$

$$\text{and} \rightarrow P_\mathrm{c} = K\rho_\mathrm{c}^{1+1/n} \tag{12}$$

inserting Eqs. (10) & (12) into Eq. (9)

$$\frac{(n+1)P_\mathrm{c}}{4\pi G\rho_\mathrm{c}^2}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -\theta^n \tag{13}$$

together with dimensionless *radial* coordinate $\xi$

$$r = r_n\,\xi \quad \text{with (const.) scale length } r_n^2 = \frac{(n+1)P_\mathrm{c}}{4\pi G\rho_\mathrm{c}^2} \tag{14}$$

our Poisson's equation (9) becomes

$\rightarrow$ so called

# The Lane-Emden Equation VI

*Lane-Emden equation* (Lane 1870; Emden 1907)

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n \tag{15}$$

with solutions "polytropes of index $n$" $\theta_n(\xi)$

Applications:

- describe i.g. self-gravitating spheres (of plasma)
- Bonnor-Ebert sphere ($n \to \infty$, so $u, e^{-u}$ instead of $\theta, \theta^n$): stable, finite-sized, finite-mass isothermal cloud with $P \neq 0$ at outer boundary $\to$ Bonnor-Ebert mass (Ebert 1955; Bonnor 1956)
- characterize (full) stellar structure models, e.g., Bestenlehner (2020) ($n = 3$, removing explicit $M_*$-dependance of $\dot{M}$-CAK desription)
- composite polytropic models for modeling of massive interstellar clouds with a hot ionized core, stellar systems with compact, massive object (BH) at centre
- generalized-piecewise polytropic EOS for NS binaries (P. Biswas 2021)

## Remarks:

if EOS is ideal gas $P = \rho N_\mathrm{A} k T/\mu$, one can get

$$P(r) = K' T^{n+1}(r), \quad T(r) = T_\mathrm{c}\theta(r) \tag{16}$$

$$\text{with } K' = \left(\frac{N_\mathrm{A} k}{\mu}\right)^{n+1} K^{-n}, \quad T_\mathrm{c} = K \rho_\mathrm{c}^{1/n} \left(\frac{N_\mathrm{A} k}{\mu}\right)^{-1} \tag{17}$$

$\rightarrow$ polytrope with EOS of ideal gas and mean molecular weight $\mu$ gives temperature profile, radial scale factor is

$$r_n^2 = \left(\frac{N_\mathrm{A} k}{\mu}\right)^2 \frac{(n+1)\,T_\mathrm{c}^2}{4\pi G P_\mathrm{c}} = \frac{(n+1) K \rho_\mathrm{c}^{1/n-1}}{4\pi G} \tag{18}$$

# The Lane-Emden Equation VIII

Requirements for physical solutions:

central density $\rho_\mathrm{c} \rightarrow \theta(\xi = 0) = 1$
spherical symmetry at center $(dP/dr|_{r=0}) \rightarrow \theta' \equiv d\theta/d\xi = 0$ at $\xi = 0 \rightarrow$ suppresses divergent solutions of the 2nd order system $\rightarrow$ regular solutions (E-solutions)
surface $P = \rho = 0 \rightarrow \theta_n = 0$ (first occurrence of that!) at $\xi_1$

## Boundary conditions for polytropic model

$\theta(0) = 1,\ \theta'(0) = 0$ at $\xi = 0$ (center)
$\theta(\xi_1) = 0$ at $\xi = \xi_1$ (surface)

So stellar radius

$$R = r_n \xi_1 = \sqrt{\frac{(n+1)P_\mathrm{c}}{4\pi G \rho_\mathrm{c}^2}}\,\xi_1 \tag{19}$$

for given $K, n$, and either $\rho_\mathrm{c}$ or $P_\mathrm{c}$ ($P_\mathrm{c} = K\rho_\mathrm{c}^{1+1/n}$)

## Analytic E-solutions

$\rightarrow$ analytic regular solutions exist for $n = 0, 1, 5$

$n = 0$ constant density sphere, $\rho(r) = \rho_\mathrm{c}$, and

$$\theta_0(\xi) = 1 - \frac{\xi^2}{6} \quad \rightarrow \xi_1 = \sqrt{6} \tag{20}$$

so $P(\xi) = P_\mathrm{c}\theta(\xi) = P_\mathrm{c}\left[1 - (\xi/\xi_1)^2\right]$. For $P_\mathrm{c}$ we need $M, R$ from Eq. (19): $P_\mathrm{c} = (3/8\pi)(GM^2/R^4)$

$n = 1$ solution $\theta_1$ is sinc function

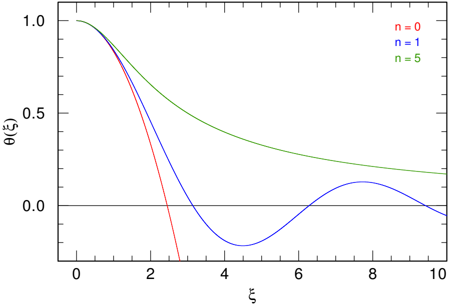
$$\theta_1 = \frac{\sin\xi}{\xi} \quad \text{with } \xi_1 = \pi \tag{21}$$

$\rightarrow \rho = \rho_\mathrm{c}\theta$ and $P = P_\mathrm{c}\theta^2$

$n = 5$ finite central density $\rho_c$ but infinite radius $\xi_1 \to \infty$ :

$$\theta_5(\xi) = \frac{1}{\sqrt{1 + \frac{\xi^2}{3}}} \tag{22}$$

contains finite mass (there is also a solution with oscillatory behavior for $\xi \to 0$, see Srivastava 1962)




$\to$ solutions with $n > 5$ have also infinite radius, but also infinite mass

# Numerical solution I

For the interesting cases $0 \le n \le 5 \to$ numerical solution

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = \frac{2}{\xi}\frac{d\theta}{d\xi} + \frac{d}{d\xi}\frac{d\theta}{d\xi} = -\theta^n \tag{23}$$

Reduction: set $x = \xi$, $y = \theta$, $z = (d\theta/d\xi) = (dy/dx)$

$$y' = \frac{dy}{dx} = z, \tag{24}$$

$$z' = \frac{dz}{dx} = -y^n - \frac{2}{x}z \tag{25}$$

Assume that we have values $y_i$, $z_i$ at a point $x_i$, so that we can get with some step size $h$: $y_{i+1}$ & $z_{i+1}$ at $x_{i+1} = x_i + h$

Then with RK4:

$$k_1 = h \cdot y'(x_i, y_i, z_i) = h \cdot (z_i) \tag{26}$$

$$\ell_1 = h \cdot z'(x_i, y_i, z_i) = h \cdot (-y_i^n - \frac{2}{x_i} z_i) \tag{27}$$

$$k_2 = h \cdot y'(x_i + h/2, y_i + k_1/2, z_i + \ell_1/2) = h \cdot (z_i + \ell_1/2) \tag{28}$$

$$\ell_2 = h \cdot z'(x_i + h/2, y_i + k_1/2, z_i + \ell_1/2) \tag{29}$$

$$= h \cdot \left( -(y_i + k_1/2)^n - \frac{2}{x_i + h/2}(z_i + \ell_1/2) \right) \tag{30}$$

$$k_3 = h \cdot y'(x_i + h/2, y_i + k_2/2, z_i + \ell_2/2) \tag{31}$$

$$\ell_3 = h \cdot z'(x_i + h/2, y_i + k_2/2, z_i + \ell_2/2) \tag{32}$$

$$k_4 = h \cdot y'(x_i + h, y_i + k_3, z_i + \ell_3) \tag{33}$$

$$\ell_4 = h \cdot z'(x_i + h, y_i + k_3, z_i + \ell_3) \tag{34}$$

$\rightarrow \quad y_{i+1} = y_i + \ldots \quad$ and $\quad z_{i+1} = z_i + \ldots$

Although $z' = -y^n - \frac{2}{x}z$ (Eq. (25)) is indeterminate for $\xi = 0$, integration can in principle be started for $\xi = 0$ for regular solutions (Cox & Giuli 1968; Hansen & Kawaler 1994) with help of power series expansion around $\xi = 0$:

$$\theta_n(\xi) = 1 - \frac{\xi^2}{6} + \frac{n}{120}\,\xi^4 - \frac{n(8n-5)}{15120}\,\xi^6 + \ldots \tag{35}$$

$$\rightarrow\ \theta_n'(\xi) = -\frac{1}{3}\xi + \frac{n}{30}\,\xi^3 - \frac{n(8n-5)}{2520}\,\xi^5 + \ldots \tag{36}$$

So for $\xi \rightarrow\ 0$ then $y' \rightarrow\ -1/3\xi = 0$.
However, better: choose $0 < \xi \ll 1$ and compute with help of Eq. (35) $y, y'(= z), z'$ (should also work for irregular solutions)

# Applying the Lane-Emden equation to stars I

construct polytropes for $n < 5$ and *given* $M$, $R$
$\rightarrow$ possible as long as $K$ not fixed
because of definition of $\theta$ from $\rho(r) = \rho_c\theta^n(r)$ (Eq. (10)) and $r = r_n\xi$ (Eq. (14))$\rightarrow dr = r_n d\xi$

$$m(r) = \int_0^r 4\pi\rho r^2 dr = 4\pi\rho_c \int_0^r \theta^n r^2 dr = 4\pi\rho_c \frac{r^3}{\xi^3}\int_0^\xi \theta^n\xi^2 d\xi \tag{37}$$

note that $r^3/\xi^3 = r_n^3$ is constant. From Lane-Emden equation (15)
$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n \rightarrow \theta^n\xi^2 = -\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)$ follows direct integration, so

$$m(r) = 4\pi\rho_c\frac{r^3}{\xi^3}\int_0^\xi -\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)d\xi = 4\pi\rho_c r^3\left(-\frac{1}{\xi}\frac{d\theta}{d\xi}\right) \tag{38}$$

$\rightarrow$ Eq. (38) contains $\xi$ and $r$, related by Eq. (14): $r/\xi = r_n = R/\xi_1$, so for the surface:

$$M = 4\pi\rho_c R^3\left(-\frac{1}{\xi}\frac{d\theta}{d\xi}\right)_{\xi=\xi_1} \tag{39}$$

# Applying the Lane-Emden equation to stars II

With help of the mean density $\overline{\rho} := M/(\frac{4}{3}\pi R^3)$ this can be written as

$$\frac{\overline{\rho}}{\rho_c} = \left(-\frac{3}{\xi}\frac{d\theta}{d\xi}\right)_{\xi=\xi_1} \tag{40}$$

Note the right hand side depends only on $n$, can be computed. E.g., for $n = 0$ $\rightarrow (-\frac{3}{\xi}\frac{d\theta}{d\xi})_{\xi=\xi_1} = 1$, and for $n = 1 \rightarrow \frac{\overline{\rho}}{\rho_c} = \frac{3}{\pi^2}$

the larger $n \rightarrow$ the smaller $\frac{\overline{\rho}}{\rho_c} \rightarrow$ the higher the density concentration

# Literature I


Bestenlehner, J. M. 2020, MNRAS, 493, 3938

Bonnor, W. B. 1956, MNRAS, 116, 351

Cox, J. P. & Giuli, R. T. 1968, Principles of stellar structure

Ebert, R. 1955, ZAp, 37, 217

Emden, R. 1907, Gaskugeln (B. Teubner, Berlin)

Hansen, C. J. & Kawaler, S. D. 1994, Stellar Interiors. Physical Principles, Structure, and Evolution. (Spinger), 84

Lane, H. J. 1870, American Journal of Science, 50, 57

Srivastava, S. 1962, ApJ, 136, 680