Exercise 9

C/C++ Solving ODEs, Lane-Emden equation, shooting method

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1. Task *Taylor method* (10 P)

Our first-order Euler method to solve an ordinary differential equation can be improved by taking higher order terms of the Taylor expansion into account. Consider the ODE

$$\frac{dx}{dt} = 2t - x = f(t, x) \tag{1}$$

with the exact solution $x(t) = e^{-t} + 2t - 2$. Find $f'(= f_t + f_x f)$, $f'' = (f_{tt} + \ldots)$ and write a C++ program that solves Eq. (1) by using:

a) the Euler method (2 P)

b) the 2nd order Taylor method (3 P)

c) the 3rd order Taylor method (4 P)

for the interval $t \in [0; 1]$ with $x_0(t_0 = 0) = -1$, using $N = 10$ subintervals. Plot the exact and the approximated solutions of all three methods. What are the errors for $x(t = 1)$ for each of the three methods (use the exact solution for comparison)? (1 P)

2. Task *Lane-Emden equation* (13 P)

Write a C++ program that computes numerically the solutions of the Lane-Emden equation for arbitrary n with help of the RK4 method. (4 P)

a) Check that your program gives the correct results by comparison with the analytic solutions (i.e., for $n = 0, 1, 5$). What are the ξ_1 for $n = 2$ and $n = 3$? (2 P)

b) Compute and plot the structure, i.e. the density profile $\theta(\xi)$, of a star (polytrope) for $n = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$. What are the total mass M_* (dimensionless) and ξ_1 for the three polytropes? (3 P)

c) Compute a polytrope for $n = 3$. With $M_* = 1 M_\odot$ and $R_* = 1 R_\odot$, i.e. for the Sun, what is the mean density $\overline{\rho}$ and what is then the central density ρ_c? Find the constant K to get the central pressure P_c. Assume an ideal gas with $\mu = 0.62$ to derive the central temperature T_c of the Sun. Compare this result with the value

   i. from proper modeling (literature) and

   ii. for the simple model of *constant* density and solving the hydrostatic equation for a *homogeneous* sphere

$$\frac{dP}{dr} = -\rho \frac{GM(r)}{r^2} \tag{2}$$

What is the central temperature T_c for $n = 2$? (4 P)

3. Task *Shooting method* (8 P)

Apply the shooting method to solve the boundary value problem

$$u''(x) = -\frac{\pi^2}{4}(u+1), \quad u(0) = 0, \quad u(1) = 1 \tag{3}$$

on the interval $x \in [0, 1]$ by combining in your program a method to solve numerically the differential equation (e.g., Euler method) and the secant method to find the root of $F(\delta) = u_\delta(1) - 1$.

Use the reduction $y_1 = u$ and $y_2 = u'$, so that

$$\frac{dy_1}{dx} = y_2 \tag{4}$$

$$\frac{dy_2}{dx} = -\frac{\pi^2}{4}(y_1 + 1) \tag{5}$$

and start with $y_{2,a}(0) = z_a = \frac{u(1)-u(0)}{1-0}$ for the first integration attempt, the result will be $F_a = y_{1,a}(1) - 1$. Then integrate again with $y_{2,b}(0) = z_b = \frac{u(1)-u(0)}{1-0} + 0.01$, this time you will get $F_b = y_{1,b}(1) - 1$. Repeat these two steps as long as $|F_b - F_a| > 10^{-6}$ by changing each time the initial guess $y_2(0) = z$ according to the secant method, i.e.,

$$z_c = z_b - F_b \cdot \frac{z_b - z_a}{F_b - F_a} \tag{6}$$

$$z_a = z_b \tag{7}$$

$$z_b = z_c \tag{8}$$

It is recommend to store $y_1(x), y_2(x)$ in an array of length $n = (1-0)/h$, with integration step size $h = 0.01$. Check your resulting solution $u(x)$ by plotting it (only every 5th integration step for better representation) together with the analytic solution which has the form

$$u(x) = a\cos(bx) + c\sin(dx) + e \tag{9}$$

with coefficients a, b, c, d, e that can be determined from the problem.