Exercise 7
C/C++ Aspects of the Kepler problem
(21.05.2025, hand in: 28.05.2025)

1. Task *The Kepler problem II* (12 P)
The program for simulating the motion of a planet around the Sun shall now be used and extended.

a) As before, create a `makefile` that executes the commands for compilation and linking when `make` is called. Implement also `make clean`.
Important note: For efficient and meaningful use of `make` you should get used to create a separate directory for each *project* and put also each function in an extra file. (1 P)

b) Modify the program by implementing also alternatively the Euler-Richardson method. How must the time step size Δt be chosen to get a *stable* orbit with the same intial values as for the Euler-Cromer method? (2 P)

c) Determine by a simple test the orbital period P (for arbitrary distances) numerically (1 P). Find P for different orbits and verify by that Kepler's 3rd law. Compare your predicted data with that for the planets of our solar system (1 P).

d) Simulate also elliptical orbits by appropriate initial values. Find the 2nd focus of the ellipse by symmetry and mark it in the graphical output. Determine the orbital period, semi-major and semi-minor axis, and verify Kepler's 3rd law also for elliptical orbits (2 P).

Also calculate the total energy for each orbit. What is the meaning of the sign of the total energy (0.5 P)? How can one check that the orbit is really an ellipse (0.5 P)?

e) Obtain the true velocity in perihelion and aphelion (1 P).
E.g., Halley's Comet (1P/Halley) has a period of about 76 years. Find by trying different values the velocity (in km/s) in the perihelion, which is at $0.59\,\mathrm{AU}$ (0.5 P). At wich distance is the aphelion (0.5 P)? Compare these distances with known orbits of planets (0.5 P).

f) Also estimate by trying at which velocity a planet with $x(t = 0) = 1\,\mathrm{AU}$ escapes from the Sun (0.5 P). Determine this velocity also for other distances from the Sun and find the function for this relation $v(r)$ (1 P).

2. Task *Perturbations* (5 extra P)
The following task shall prove your intuitive understanding of Newton's laws of motion.

a) We want to analyze the impact of small perturbations on the orbit. Therefore we will give the planet during its motion a *kick* (additional momentum) into

i. radial or

ii. tangential

direction. Try to estimate in advance how the shape of the orbit will change for both cases. Describe qualitatively the resulting orbit (1 P).

b) The change of momentum shall be applied with help of the mouse (cursor).

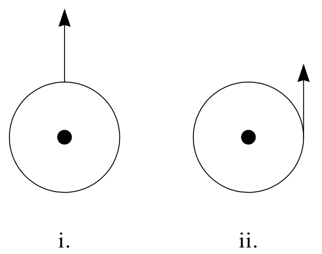

Figure 1: The momentum in y direction will be applied in radial (i.) or tangential (ii.) direction, depending on the position.

The position of the mouse at a *button press* is obtained within the event loop via `WGetMousePos(...)`[1]. When the planet approaches these coordinates the next time, a kick in y direction is applied once(!) in the function `euler`. The applied change in the velocity (= momentum/m) in y direction should be 10% of the *absolut value* of the current velocity. (2 P)

c) Is this new orbit stable? (1 P)

d) Maybe you will find the change of the trajectory counter-intuitive. How can the change of the trajectory be understood with help of Newton's 2nd law? (1 P)

3. Task *Alternative laws of gravitation* (5 P)

a) Consider the case of a modified gravitational force which is proportional to $Cm/r^{2+\epsilon}$, where $\epsilon \ll 1$. Use again astronomical units with $C \equiv 4\pi^2$. Use the Euler-Richardson method. (2 P)
Choose $\epsilon = 0.05$ and the following initial values: $x(t=0) = 1$, $y(t=0) = 0$, $v_x(t=0) = 0$, $v_y(t=0) = 5$. Follow the planet over several periods. How does the orbit look like (0.5 P)? Make sure that the result does not depend on the value of Δt.
How does the change of the orbit per period depend on the size of the semi-major axis (qualitatively)? (0.5 P) Determine the influence of ϵ on the result.
What could be the practical meaning of such a law of gravitation (1 P)? (*Hint: orbit of Mercury*)

b) Implement analogously a law of gravitation for which the force is $\propto 1/r^3$. How must the start velocity $v_y(t=0)$ be chosen to get a circular orbit (analytic formula)? Adapt Δt if necessary so that a stable circular orbit is obtained. (1 P)

c) *Bonus:* Apply again a small kick on the planet from task 3b as in task 2, this time with $0.01|v|$. Is the orbit stable against perturbations? (1 extra P)

4. Task *A binary star* (3 extra P)

Modify your program to simulate the motion of a planet around a binary star. For simplicity, both stars should have the same mass and fixed positions (i.e. we're in the co-moving frame of their orbital motion and neglect any effect of this motion): One star is at $(0; 0)$, the other one is at $(2; 0)$.
So, you only have to modify the function `euler()` such that for the calculation of accelerations acting on the planet the forces of both stars are taken into account.
The planet starts at $(1.1; 1)$. Vary the x and y components of the start velocity so that you get different kinds of trajectories. Describe them qualitatively.
Also try to obtain a stable orbit. What is the general trick?

[1]Note: In the modified version of `Xgraphics` the data type of `x` and `y` is `double *` and *not* `float *`.