The Eddington limit in rotating massive stars

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1. Introduction: LBVs and the Eddington limit

The most spectacular phenomenon associated with Luminous Blue Variables (LBVs) are certainly their giant eruptions, which can last several years and may be recurrent on a time scale of centuries (cf. Bohannan, this volume; Conti, this volume), η Carinae's A.D. 1840–1860 and P Cygni's A.D. 1600 outbursts being the major examples. Circumstellar shells, which exist around virtually all LBVs (Nota et al. 1995), may be considered as fossil records of such outbursts (García-Segura et al. 1996). No fully selfconsistent models for the giant eruptions do yet exist; however, among other ideas (e.g., Glatzel, this volume; Guzik et al., this volume), the evolution of massive stars towards the Eddington limit has been put forward as an explanation by various authors (cf. Humphreys & Davidson 1994, and references therein). In the present paper we show that, in case the Eddington limit is involved in LBV eruptions, in fact not the Eddington limit itself but rather critical rotation defines the hydrostatic stability limit, no matter what the actual rotation rate of the star is.

2. Does the Eddington limit apply in the stellar interior?

The Eddington limit, in brief, is a limit to hydrostatic stability defined by the condition that the outwards directed force induced by the radiation momentum balances the inwards directed gravity. We want to emphasize that, as a stability limit, the Eddington limit applies only to hydrostatic situations. This does not mean that radiation forces are unimportant in hydrodynamic situations (e.g stellar pulsations, cf. Cox et al., this volume); however, a stability limit makes of course no sense in an unstable situation.

The "classical Eddington limit" $L > L_{\rm edd} := 4\pi cGM/\kappa_{\rm e}$ (cf. Lamers, this volume), with L and M being stellar luminosity and mass, and $\kappa_{\rm e}$ the electron scattering opacity, is derived from the momentum equation as $a_{\rm rad}+g > 0$, where $a_{\rm rad} = \kappa_{\rm e}L/(4\pi R^2 c)$ is the radiative acceleration and the gravity $g = GM/R^2$, with R being the stellar radius. Instead of the above definition of the Eddington limit, one may introduce the dimensionless Eddington factor $\Gamma := L/L_{\rm edd}$, and by replacing $\kappa_{\rm e}$ by the true flux-mean opacity coefficient κ one gets

$$\Gamma = \frac{\kappa}{4\pi cG}\frac{L}{M} \tag{1}$$

and instability for $\Gamma > 1$.

Eq.(1) can be generalized to be evaluated in the stellar interior, i.e. $\Gamma(r) = \frac{\kappa(r)}{4\pi cG}\frac{L(r)}{M(r)}$, and some authors used the condition $\Gamma(r) > 1$ as stability criterion. However, this is wrong since for $\Gamma(r) \rightarrow 1$ convection must set in. This can be seen for the case of LBVs, where the equation of state of gas and radiation applies as $P := P_{\rm gas} + P_{\rm rad} := \frac{\Re}{\mu}\rho T + \frac{a}{3}T^4$, even when we neglect ionization which would reduce the adiabatic temperature gradient $\nabla_{\rm ad}$ and make convection more likely. With $\beta := P_{\rm gas}/P$, we have

$$\nabla_{\rm ad} = \frac{8 - 6\beta}{32 - 24\beta - 3\beta^2} \tag{2}$$

(Kippenhahn & Weigert 1990). The radiative temperature gradient $\nabla_{\rm rad} := \frac{3}{16\pi acG}\frac{\kappa(r)L(r)P}{M(r)T^4}$ can be written as

$$\nabla_{\rm rad} = \frac{\Gamma(r)}{4\,(1-\beta)}. \tag{3}$$

The Schwarzschild criterion for convection $\nabla_{\rm rad} \geq \nabla_{\rm ad}$ can thus be expressed as

$$\Gamma(r) \geq (1-\beta)\frac{32 - 24\beta}{32 - 24\beta - 3\beta^2}. \tag{4}$$

Since, for $\beta > 0$, the r.h.s. of (4) is always smaller than 1, the Schwarzschild criterion is fulfilled for $\Gamma(r) \rightarrow 1$.

Since the convective flux does not contribute to the radiative acceleration one might think of a modified Eddington limit in the stellar interior according to $\Gamma'(r) := \frac{\kappa(r)}{4\pi cG}\frac{L(r) - L_{\rm conv}(r)}{M(r)} > 1$, where the convective luminosity $L_{\rm conv}(r)$ is subtracted. Especially in the deep stellar interior where convection is almost adiabatic, almost arbitrarily large luminosities can be transported through convection. Therefore, even for $\Gamma(r) \gg 1$, $\Gamma'(r) < 1$ is the result and no instability occurs.

However, even for convection zones close to the stellar surface, where convection becomes strongly non-adiabatic and $L_{\rm conv} \rightarrow 0$, $\Gamma'(r) = \Gamma(r) > 1$ is *not* a criterion for instability. This can be seen when the Schwarzschild criterion is written in entropy formulation as $\frac{ds}{dr} \leq 0$. Since $\frac{ds}{dr} \propto \frac{1}{a^2}\frac{dP}{dr} - \frac{d\rho}{dr}$ (with a being the adiabatic sound speed), we have for adiabatic convection ($\frac{ds}{dr} = 0$) $\frac{d\rho}{dr} = \frac{1}{a^2}\frac{dP}{dr}$. As in a hydrostatic situation the total pressure decreases outwards, the density must do the same. But for strongly non-adiabatic convection $\frac{ds}{dr}$ is large and negative, and thus a *positive* density gradient may be established. This leads to a positive gas pressure gradient, which can stabilize the stratification even for $\Gamma'(r) > 1$.

In summary, the Eddington limit, in whichever form, can not be applied in the stellar interior to predict instability. It only applies at the stellar surface, where density inversions can not occur and the convective flux is negligible. In the next section, however, we show that the Eddington limit by itself should also not applied at the stellar surface.

3. The Ω-limit

In a hydrostatic rotating star, the momentum equation has to include the centrifugal acceleration $a'_{\rm cen}(r,\theta) = a_{\rm cen}(r)cos(\theta)$, where θ is the latitudal coordinate. To obtain a stability criterion, we follow Section 2 and only evaluate the momentum equation at the stellar surface, where density inversions and convective flux contributions are negligible. Furthermore, we concentrate on the equatorial plane, since there the destabilizing centrifugal term is largest. Defining $v_{\rm rot}^2/R := a_{\rm cen}(R,0^\circ)$, we are left with a momentum balance as $a_{\rm rad} + g + v_{\rm rot}^2/R = 0$ or $\frac{1}{R}\left(\frac{\kappa L}{4\pi Rc} + \frac{GM}{R} + v_{\rm rot}^2\right) = 0$. We can thus define

$$v_{\rm crit}^2 := \frac{GM}{R}(1-\Gamma) \tag{5}$$

and obtain as criterion for instability

$$\Omega := \frac{v_{\rm rot}}{v_{\rm crit}} > 1. \tag{6}$$

It is crucial to realize that for $\Gamma \rightarrow 1$ Eq. (5) gives $v_{\rm crit} \rightarrow 0$. Therefore, if we assume a star to evolve towards the Eddington limit ($\Gamma \rightarrow 1$), no matter what its rotation rate may be, it will arrive at critical rotation well *before* $\Gamma = 1$ is actually reached. Therefore, one may rather speak of the "Ω-limit" instead of the Eddington limit.

As a consequence, the location of the occurrence of the instability in the HR diagram depends on the stellar rotation rate. In contrast to the Eddington limit, the Ω-limit is not — for a fixed metallicity and L/M-relation — a single line in the HR diagram. Furthermore, while the most massive stars may arrive at the Ω-limit independent of their rotation speed, it may happen that two less massive single stars start with the same initial mass and metallicity, but only the faster rotating one arrives at the Ω-limit while the slower one does not and evolves further into the red supergiant region. This scenario might apply to the less luminous LBVs, which do have red supergiant counterparts of equal luminosity.

A second consequence is that the outflow which occurs due to a star hitting the Ω-limit is predicted to be highly bipolar. This is illustrated by a hydrodynamical simulation of the giant eruption of η Car by García-Segura et al. (this volume). The fact that virtually all LBV nebulae are highly bipolar supports the idea that the Ω-limit is actually involved in the LBV instability.

4. The evolution of rotating massive stars

In the following, we shall describe several effects which may happen when massive stars approach the Ω-limit. As we shall see, it must be beyond the scope of this work to investigate these effects quantitatively. However, striking qualitative features can be derived from the simplified models of rotating, mass losing $60\,M_\odot$ stars presented below.

In calculating these models, we assume that the stars remain always rigidly rotating, which is well justified given the short time scales of angular momentum redistribution (Zahn 1992, Maeder 1997), and confirmed by more sophisticated models (Fliegner et al. 1996). The angular momentum is carried only as a passive quantity, i.e. we neglect any feedback of rotation on the stellar structure. For stars approaching the Ω-limit, this may appear oversimplified; however, note that Ω becomes close to unity only in a tiny fraction of the the stellar mass due to the iron opacity peak, while it remains small in the rest of the star which always remains perfectly spherical. Much more uncertain is the evaluation of the Eddington factor Γ. In the models presented here, we adopted the maximum value occurring in the gray plane-parallel atmosphere down to an optical depth of $\tau = 100$; thereby the convective flux and density inversions can be neglected. However, when Ω approaches 1, these layers may expand in the equatorial region of the star. Since the expansion will lead to cooling, the opacity in these layers will increase. Furthermore, viscous coupling would increase their specific angular momentum and bring them even closer the Ω-limit. Therefore, our values of Γ and Ω may in fact be lower limits.

In the non-rotating case, the value of Γ evaluated as described above is 0.78 on the zero age main sequence ($\log T_{\rm eff} = 50\,000\,{\rm K}$) and rises to 0.96 at $t = 2.6\,10^6$ yr ($\log T_{\rm eff} = 25\,000\,{\rm K}$) due to the iron opacity peak. The core H-burning phase is finished at $t = 3.4\,10^6$ yr and $\log T_{\rm eff} = 20\,000\,{\rm K}$ with $\Gamma = 0.84$. The evolution of Γ is reflected in the behavior of the critical rotational velocity (cf. Eq. 5) as shown in Fig. 1; it has a pronounced minimum of $v_{\rm crit,min} \simeq 100\,{\rm km\,s^{-1}}$ at $t \simeq 2.6\,10^6$ yr. Since the rotational velocity tends to remain almost constant on the main sequence (Fig. 1) the question arises: what happens to a star with an initial rotational velocity above $\sim 100\,{\rm km\,s^{-1}}$?

As shown by Friend & Abbott (1986), rotating massive stars have an enhanced radiation driven stellar wind mass loss well before they actually hit the Ω-limit, with an enhancement factor

$$\frac{\dot M}{\dot M(v_{\rm rot}=0)} = \left(\frac{1}{1-\Omega}\right)^{\xi}, \quad \xi \simeq 0.43. \tag{7}$$

Formally, $\dot M \to \infty$ for $\Omega \to 1$ in Eq. (7). However, the mass loss remains finite due to the effect that it imposes a loss of angular momentum which results in a (non-magnetic) spin-down of the star: In rigidly rotating stars, the specific angular momentum increases from the center to the surface; due to mass loss, it is continuously transported from the core to the envelope where it is blown off. Therefore, the mass loss rate (due to Eq. 7) of the considered stars at the Ω-limit, $\dot M_\Omega$, is self-regulated by the condition $\dot M_\Omega = \dot M(v_{\rm rot} = v_{\rm crit})$.

It can be seen from Fig. 1 that $\dot M_\Omega \simeq 10^{-5}\ M_\odot\ {\rm yr}^{-1}$ for the considered $60\,M_\odot$ sequences. It should be stressed that, although our model is simplified and our analysis only qualitative, the derived mass loss rate at the Ω-limit is expected to be even quantitatively correct, since it is established by the rate of angular momentum loss which is associated with the mass loss. Once the critical rotational velocity is reached, the angular momentum loss rate is set by the expansion time scale of the star (i.e. its speed in the HR diagram) but does not depend on the value of the critical velocity.

A further striking feature shown in Fig. 1 is the convergence of the rotational velocities to the value of $v_{\rm crit,min}$ (i.e., $\simeq 100\,{\rm km\,s^{-1}}$), which is produced by

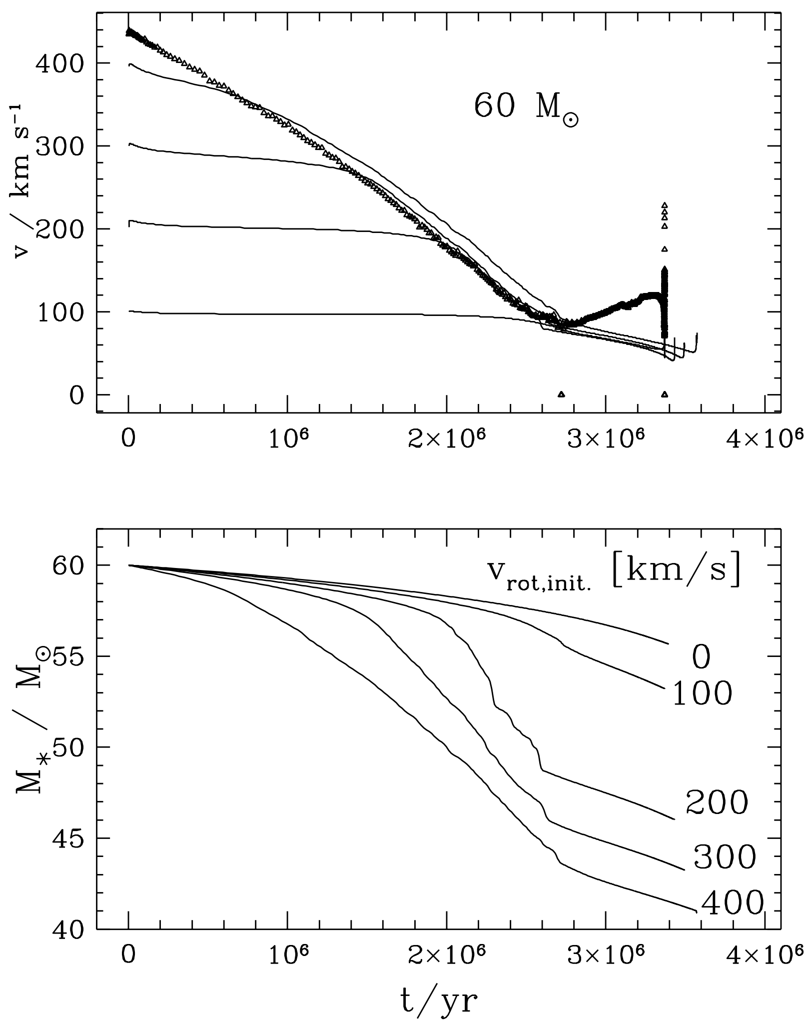

Figure 1. **Upper panel:** Evolution of the equatorial rotation velocity with time during the core hydrogen burning phase of four $60\,M_{\odot}$ sequences with different initial rotation rates (see at $t = 0$). The evolution of the critical rotation velocity (Eq. 5) is displayed for the sequence with $v_{\rm rot,init.} = 100\,{\rm km\,s^{-1}}$ by the triangles. It is very similar for the other sequences. For $v_{\rm crit} \simeq v_{\rm rot}$, the stars evolve at the Ω-limit. **Lower panel:** Evolution of the stellar mass with time for the same $60M_{\odot}$ sequences. The initial equatorial rotation velocities are given as labels. For comparison, the evolution of a non-rotating star is shown in addition.

the fact that the time dependence of the critical rotational velocity is almost independent of the mass loss history, in particular its minimum value. Note, however, that although the rotation velocities of the four sequences displayed in Fig. 1 at the end of the main sequence evolution are almost identical, their masses at that time are greatly different, and thus their ensuing post-main sequence evolution is expected to be very different as well.

Our main sequence models at the Ω-limit can certainly not be related to LBVs, but perhaps rather to B[e] stars; the material of stars at the Ω-limit which has excess angular momentum can not remain on the star but it is also not clear whether it can be pushed to infinity; some of it may form a ring or disk close to the star. However, as the mass loss rate at the Ω-limit depends on the evolutionary speed of the star in the HR diagram, one can expect values of the order $10^{-5}\ M_{\odot}\ \mathrm{yr}^{-1} \times \tau_{\mathrm{H}}/\tau_{\mathrm{KH}}$ for the expansion phase after core H-exhaustion, when the star hits the Ω-limit again due to the helium opacity peak. Here, τ_{H} is the H-burning time scale and τ_{KH} the thermal time scale of the star, such that values of $\dot{M}_{\Omega}$ of the order of $10^{-3} - 10^{-2}\ M_{\odot}\ \mathrm{yr}^{-1}$ can be expected, which is the order of magnitude observed for LBVs.

5. Summary

The following conclusions emerge:

- The Eddington limit does not apply in the stellar interior.
- At the stellar surface, the Eddington limit should be replaced by the Ω-limit (Eqs. 5 and 6).
- Massive stars may reach the Ω-limit already on the main sequence (Fig. 1).
- Massive stars approaching the Ω-limit have enhanced stellar wind mass loss (Friend & Abbott 1986, Eq. 7).
- The mass loss rate at the Ω-limit is determined through the corresponding angular momentum loss rather than through atomic physics.
- The expected mass loss of post-main sequence stars at the Ω-limit corresponds to values found for LBVs.
- The bipolarity of LBV nebulae supports the idea of the Ω-limit being involved in the major LBV eruptions.
- The initial angular momentum is a third independent initial parameter for the evolution of massive stars, as important as initial mass and metallicity.

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References

Fliegner J., Langer N., Venn K. 1996, A&A, 308, L13

Friend D.B., Abbott D.C. 1986, Ap. J. , 311, 701

García-Segura G., Mac Low M.-M., Langer N. 1996, A&A, 305, 299

Humphreys R.M., Davidson K. 1994, PASP, 106, 1025

Kippenhahn R., Weigert A. 1990, Stellar Structure and Evolution, Springer
Lamers H.J.G.L.M., Leitherer C. 1993, Ap. J. , 412, 771
Langer N. 1992, A&A, 265, L17
Maeder A. 1997. A&A, in press
Nota A., Livio M., Clampin M., Schulte-Ladbeck R. 1995, Ap. J. , 448, 788
Zahn, J.P. 1992, A&A, 265, 115