Computational Astrophysics I: Introduction and basic concepts

Helge Todt

Astrophysics
Institute of Physics and Astronomy
University of Potsdam

SoSe 2020
The two-body problem
We remember (?):

The Kepler’s laws of planetary motion (1619)

1. Each planet moves in an elliptical orbit where the Sun is at one of the foci of the ellipse.
2. The velocity of a planet increases with decreasing distance to the Sun such, that the planet sweeps out equal areas in equal times.
3. The ratio $P^2/a^3$ is the same for all planets orbiting the Sun, where $P$ is the orbital period and $a$ is the semimajor axis of the ellipse.

The 1. and 3. Kepler’s law describe the shape of the orbit (Copernicus: circles), but not the time dependence $\vec{r}(t)$. This can in general not be expressed by elementary mathematical functions (see below). Therefore we will try to find a *numerical* solution.
Equations of motion II

Earth-Sun system

→ two-body problem → one-body problem via reduced mass of lighter body (partition of motion):

\[ \mu = \frac{M m}{m + M} = \frac{m}{M} + 1 \]  \hspace{1cm} (1)

as \( m_E \ll M_\odot \) is \( \mu \approx m \), i.e. motion is relative to the center of mass \( \equiv \) only motion of \( m \). Set \((0, 0)\) to the source of the force field of \( M \).

Moreover: Newton’s 2. law:

\[ m \frac{d^2 \vec{r}}{dt^2} = \vec{F} \]  \hspace{1cm} (2)

and force field according to Newton’s law of gravitation:

\[ \vec{F} = -\frac{GMm}{r^3} \vec{r} \]  \hspace{1cm} (3)
Kepler’s laws, as well as the assumption of a central force imply conservation of angular momentum → motion is only in a plane (→ Kepler’s 1st law).

So, we use Cartesian coordinates in the $xy$-plane:

\[ F_x = -\frac{GMm}{r^3} x \]
\[ F_y = -\frac{GMm}{r^3} y \]

The equations of motion are then:

\[ \frac{d^2 x}{dt^2} = -\frac{GM}{r^3} x \]
\[ \frac{d^2 y}{dt^2} = -\frac{GM}{r^3} y \]

where \( r = \sqrt{x^2 + y^2} \)
To derive the analytic solution for equation of motion $\vec{r}(t) \rightarrow$ use polar coordinates: $\phi, r$

1. use conservation of angular momentum:

$$\mu r^2 \dot{\phi} = \ell = \text{const.} \quad (9)$$

$$\dot{\phi} = \frac{\ell}{\mu r^2} \quad (10)$$

2. use conservation of energy:

$$E = \frac{1}{2} \mu \dot{r}^2 + \frac{\ell^2}{2\mu r} - \frac{GM\mu}{r} \quad (11)$$

$$\dot{r}^2 = \frac{2E}{\mu} - \frac{\ell^2}{\mu^2 r^2} + \frac{2GM}{r} \quad (12)$$

→ two coupled equations for $r$ and $\phi$
decouple Eq. (10), use the orbit equation $r = \frac{\alpha}{1 + e \cos \phi}$ with numeric eccentricity $e$ and $\alpha \equiv \frac{\ell^2}{GM\mu^2}$ gives separable equation for $\dot{\phi}$

$$\dot{\phi} = \frac{d\phi}{dt} = \frac{G^2 M^2 \mu^3}{\ell^3} (1 + e \cos \phi)^2$$  \hspace{1cm} (13)

$$t = \int dt' = k \int \frac{d\phi'}{(1 + e \cos \phi')^2} = f(\phi)$$  \hspace{1cm} (14)

right-hand side integral can be looked up in, e.g., Bronstein:

$$= \frac{e \sin \phi}{(e^2 - 1)(1 + e \cos \phi)} - \frac{1}{e^2 - 1} \int \frac{d\phi}{1 + e \cos \phi}$$ \hspace{1cm} (15)

$\rightarrow e \neq 1$: parabola excluded; the integral can be further simplified:

$$0 \leq e < 1 : \int \frac{d\phi}{1 + e \cos \phi} = \frac{2}{\sqrt{1 - e^2}} \arctan \left( \frac{(1 - e) \tan \frac{\phi}{2}}{\sqrt{1 - e^2}} \right)$$  \hspace{1cm} (16)
for the ellipse; and for the hyperbola:

\[ e < 1 : \int \frac{d\phi}{1 + e \cos \phi} = \frac{1}{\sqrt{e^2 - 1}} \ln \frac{(e - 1) \tan \frac{\phi}{2} + \sqrt{e^2 - 1}}{(e - 1) \tan \frac{\phi}{2} - \sqrt{e^2 - 1}} \]  

\[ \text{Eqn. (16) \& (17) must be inverted to get } \phi ! \]

\[ \rightarrow \text{only easy for } e = 0 \rightarrow \text{circular orbit} \]

\[ t = k \int d\phi' = k \phi \rightarrow \phi(t) = k^{-1} t = \frac{G^2 M^2 \mu^3}{\ell^3} t \]  

and from orbit equation \( r = \alpha = \frac{\ell^2}{GM\mu^2} = \text{const.} \)

For the general case, it is much easier to solve the equations of motion numerically.
Alternative formulation for time dependency in case of ellipse ($0 \leq e < 1$):

Consider an angle $\psi$ (or $E$, eccentric anomaly) defined by $\angle \Pi QO$.
position \((r, \phi)\) of the body \(P\) can be described in terms of \(\psi\):

\[
x_P = r \cos \phi = a \cos \psi - ae
\]  
(19)

\[
y_P = r \sin \phi = a \sin \psi \sqrt{1 - e^2}
\]  
(20)

(with \(\frac{PR}{QR} = \frac{b}{a} = \sqrt{1 - e^2}\)), square both equations and add them up:

\[
r = a(1 - e \cos \psi)
\]  
(21)

Now, to find \(\psi = \psi(t)\), need relationship between \(d\phi\) and \(d\psi\), so combine Eqn. (20) & (21)

\[
\sin \phi = \frac{b \sin \psi}{a(1 - e \cos \psi)} \quad |d/dx'
\]  
(22)

\[
\cos \phi d\phi = \frac{b}{a} \frac{(\cos \psi(1 - e \cos \psi)d\psi - e \sin^2 \psi d\psi)}{(1 - e \cos \psi)^2}
\]  
(23)

\[
d\phi = \frac{b}{a(1 - e \cos \psi)} d\psi
\]  
(24)
Excursus: The Kepler equation III

together with the angular momentum \(d\phi = \frac{\ell}{\mu r^2} dt\):

\[
(1 - e \cos \psi) d\psi = \frac{\ell}{\mu ab} dt
\]

\[
= \text{set } t = 0 \rightarrow \psi(0) = 0, \text{ integration:}
\]

\[
\psi - e \sin \psi = \frac{\ell t}{\mu ab}
\]

use Kepler’s 2nd law \(\frac{\pi ab}{P} = \frac{\ell}{2\mu}\) with \(\pi ab\) the area of the ellipse, we get

\[
\ell/(\mu ab) = 2\pi/P \equiv \omega \text{ (orbital angular frequency), so:}
\]

Kepler’s equation

\[
\psi - e \sin \psi = \omega t
\]

\[
E - e \sin E = M \quad \text{(astronomer’s version)}
\]

\(M\): mean anomaly = angle for constant angular velocity
Kepler’s equation \( E(t) - e \sin E(t) = M(t) \)

- is a transcendental equation
- can be solved by, e.g., Newton’s method
- because of \( E = M + e \sin E \), also (Banach) fixed-point iteration possible (slow, but stable), already used by Kepler (1621):

\[
E = M ; \\
\text{for} \ (\text{int} \ i = 0 \ ; \ i < n \ ; \ ++i) \\
\quad E = M + e * \sin(E) ;
\]

- can be solved, e.g., by Fourier series \( \rightarrow \) Bessel (1784-1846):

\[
E = M + \sum_{n=1}^{\infty} \frac{2}{n} J_n(ne) \sin(nM) \quad (30)
\]

\[
J_n(ne) = \frac{1}{\pi} \int_{0}^{\pi} \cos(nx - ne \sin x) \, dx \quad (31)
\]
Circular orbits

A special case as a solution of the equations of motion (6) & (7) is the circular orbit. Then:

\[ \ddot{r} = \frac{v^2}{r} \]  \hspace{1cm} (32)

\[ \frac{mv^2}{r} = \frac{GMm}{r^2} \] \hspace{1cm} (equilibrium of forces)  \hspace{1cm} (33)

\[ \Rightarrow v = \sqrt{\frac{GM}{r}} \]  \hspace{1cm} (34)

The relation (34) is therefore the condition for a circular orbit. Moreover, Eq. (34) yields together with

\[ P = \frac{2\pi r}{v} \]  \hspace{1cm} (35)

\[ \Rightarrow P^2 = \frac{4\pi^2}{GM} r^3 \]  \hspace{1cm} (36)
Astronomical units

For our solar system it is useful to use astronomical units (AU):

\[ 1 \text{AU} = 1.496 \times 10^{11} \text{m} \]

and the unit of time is the (Earth-) year

\[ 1 \text{a} = 3.156 \times 10^7 \text{s} \quad (\approx \pi \times 10^7 \text{s}), \]

so, for the Earth \( P = 1 \text{a} \) and \( r = 1 \text{AU} \)

Therefore it follows from Eq. (36):

\[
GM = \frac{4\pi^2 r^3}{P^2} = 4\pi^2 \text{AU}^3 \text{a}^{-2}
\]  

(37)

i.e. we set \( GM \equiv 4\pi^2 \) in our calculations.

Advantage: handy numbers!

Thus, e.g. \( r = 2 \) is approx. \( 3 \times 10^{11} \text{m} \) and \( t = 0.1 \) corresponds to \( 3.16 \times 10^6 \text{s} \), and \( v = 6.28 \) is roughly 30 km/s.
The equations of motion (6) & (7):

\[ \frac{d^2 \vec{r}}{dt^2} = -\frac{GM}{r^3} \vec{r} \]  

are a system of differential equations of 2nd order, that we shall solve now. Formally: \textit{integration} of the equations of motion to obtain the \textit{trajectory} \( \vec{r}(t) \).

\begin{align*}
\text{Step 1: reduction} \\
\text{Rewrite Newton’s equations of motion as a system of differential equations of 1st order (here: 1d):} \\
\quad v(t) &= \frac{dx(t)}{dt} \quad \& \quad a(t) = \frac{dv(t)}{dt} = \frac{F(x, v, t)}{m} \quad (39)
\end{align*}
Step 2: Solving the differential equation

Differential equations of the form (initial value problem)

\[ \frac{dx}{dt} = f(x, t), \quad x(t_0) = x_0 \]  \hspace{1cm} (40)

can be solved numerically (discretization\(^1\)) by as simple method:

Explicit Euler method (“Euler’s polygonal chain method”)

1. choose step size \(\Delta t > 0\), so that \(t_n = t_0 + n\Delta t\), \(n = 0, 1, 2, \ldots\)
2. calculate the values (iteration):
   \[ x_{n+1} = x_n + f(x_n, t_n)\Delta t \]

Obvious: The smaller the step size \(\Delta t\), the more steps are necessary, but also the more accurate is the result.

\(^1\)I.e. we change from calculus to algebra, which can be solved by computers.
Why “polygonal chain method”? 

Exact solution (–) and numerical solution (–).
Derivation from the Fundamental theorem of calculus

Integration of the ODE \( \frac{dx}{dt} = f(x, t) \) from \( t_0 \) till \( t_0 + \Delta t \) (41)

\[
\int_{t_0}^{t_0+\Delta t} \frac{dx}{dt} \, dt = \int_{t_0}^{t_0+\Delta t} f(x, t) \, dt \quad (42)
\]

\[\Rightarrow x(t_0 + \Delta t) - x(t_0) = \int_{t_0}^{t_0+\Delta t} f(x(t), t) \, dt \quad (43)\]

Apply rectangle method for the integral:

\[
\int_{t_0}^{t_0+\Delta t} f(x(t), t) \, dt \approx \Delta t f(x(t_0), t_0) \quad (44)
\]

Equating (43) with (44) yields Euler step

\[x(t_0 + \Delta t) = x(t_0) + \Delta t f(x(t_0), t_0) \quad (45)\]
Derivation from Taylor expansion

\[ x(t_0 + \Delta t) = x(t_0) + \Delta t \frac{dx}{dt}(t_0) + O(\Delta t^2) \]  
(46)

use \( \frac{dx}{dt} = f(x, t) \)  
(47)

\[ x(t_0 + \Delta t) = x(t_0) + \Delta t f(x(t_0), t_0) \]  
(48)

while neglecting term of higher order in \( \Delta t \)
For the system (39)

\[ v(t) = \frac{dx(t)}{dt} \quad \& \quad a(t) = \frac{dv(t)}{dt} = \frac{F(x, v, t)}{m} \]

this means

**Euler method for solving Newton’s equations of motion**

\[ v_{n+1} = v_n + a_n \Delta t = v_n + a_n(x_n, t) \Delta t \]  
\[ x_{n+1} = x_n + v_n \Delta t \]  

(49)  
(50)

We note:

- the velocity at the end of the time interval \( v_{n+1} \) is calculated from \( a_n \), which is the acceleration at the beginning of the time interval
- analogously \( x_{n+1} \) is calculated from \( v_n \)
Example: Harmonic oscillator

```cpp
#include <iostream>
#include <cmath>
using namespace std;

int main () {
    int n = 10001, nout = 500;
    double t, v, v_old, x;
    double const dt = 2. * M_PI / double(n-1);
    x = 1. ; t = 0. ; v = 0. ;
    for (int i = 0 ; i < n ; ++i) {
        t = t + dt ; v_old = v ;
        v = v - x * dt ;
        x = x + v_old * dt ;
        if (i % nout == 0) // print out only each nout step
            cout << t << " " << x << " " << v << endl ;
    }
    return 0 ;
}
```
We will slightly modify the explicit Euler method, but such that we obtain the same differential equations for $\Delta t \to 0$.
For this new method we use $v_{n+1}$ for calculating $x_{n+1}$:

**Euler-Cromer method (semi-implicit Euler method)**

\[
\begin{align*}
v_{n+1} &= v_n + a_n \Delta t \quad \text{(as for Euler)} \\
x_{n+1} &= x_n + v_{n+1} \Delta t
\end{align*}
\] (51) (52)

Advantage of this method:
- as for Euler method, $x$, $v$ need to be calculated only once per step
- especially appropriate for oscillating solutions, as energy is conserved much better (see below)
Proof of stability (Cromer 1981):

\[ v_{n+1} = v_n + F_n \Delta t \quad (= v_n + a(x_n) \Delta t, \ m = 1) \quad (53) \]
\[ x_{n+1} = x_n + v_{n+1} \Delta t \quad (54) \]

Without loss of generality, let \( v_0 = 0 \). Iterate Eq. (53) \( n \) times:

\[ v_n = (F_0 + F_1 + \ldots + F_{n-1}) \Delta t = S_{n-1} \quad (55) \]
\[ x_{n+1} = x_n + S_n \Delta t \quad (56) \]

\[ S_n := \Delta t \sum_{j=0}^{n} F_j \quad (57) \]

Note that for explicit Euler Eq. (56) is \( x_{n+1} = x_n + S_{n-1} \Delta t \).
The change in the kinetic energy $K$ between $t_0 = 0$ and $t_n = n\Delta t$ is because of Eq. (53) and $v_0 = 0$

$$\Delta K_n = K_n - K_0 = K_n = \frac{1}{2} S_{n-1}^2 \tag{58}$$

The change in the potential energy $U$:

$$\Delta U_n = -\int_{x_0}^{x_n} F(x) dx \tag{59}$$
Now use the trapezoid rule for this integral

\[ \Delta U_n = -\frac{1}{2} \sum_{i=0}^{n-1} (F_i + F_{i+1})(x_{i+1} - x_i) \]  

(60)

\[ = -\frac{1}{2} \Delta t \sum_{i=0}^{n-1} (F_i + F_{i+1})S_i \quad (\rightarrow \text{Eq. 54}) \]  

(61)

\[ = -\frac{1}{2} \Delta t^2 \sum_{i=0}^{n-1} \sum_{j=0}^{i} (F_i + F_{i+1})F_j \quad (\rightarrow \text{Eq. 57}) \]  

(62)
As $j$ runs from 0 to $i \to \Delta U_n$ has same squared terms as $\Delta K_n$, see:

\[
\Delta U_n = -\frac{1}{2} \Delta t^2 \left( \sum_{i=0}^{n-1} F_i^2 + \sum_{i=0}^{n-1} \sum_{j=0}^{i-1} F_i F_j + \sum_{i=1}^{n} \sum_{j=0}^{i-1} F_i F_j \right) 
\]

(63)

\[
= -\frac{1}{2} \Delta t^2 \left( \sum_{i=0}^{n-1} F_i^2 + 2 \sum_{i=0}^{n-1} \sum_{j=0}^{i-1} F_i F_j + F_n \sum_{j=0}^{i-1} F_j \right) 
\]

(64)

\[
= -\frac{1}{2} S_n^2 - \frac{1}{2} \Delta t F_n S_{n-1} 
\]

(65)

Hence the total energy changes as

\[
\Delta E_n = \Delta K_n + \Delta U_n = \frac{1}{2} S_n^2 - \frac{1}{2} S_{n-1}^2 - \frac{1}{2} \Delta t F_n S_{n-1} 
\]

(66)

\[
= -\frac{1}{2} \Delta t F_n S_{n-1} = -\frac{1}{2} \Delta t F_n v_n 
\]

(67)
For oscillatory motion: $v_n = 0$ at turning points, $F_n = 0$ at equilibrium points $\rightarrow \Delta E_n = -\frac{1}{2} \Delta t F_n v_n$ is 0 four times of each cycle $\rightarrow \Delta E_n$ oscillates with $T/2$.

As $F_n$ and $v_n$ are bound $\rightarrow \Delta E_n$ is bound, more important: average of $\Delta E_n$ over half a cycle ($T$)

$$
\langle \Delta E_n \rangle = \Delta t^2 \frac{1}{2} \frac{T}{\Delta t} \sum_{n=0}^{T/\Delta t} F_n v_n \approx \Delta t \frac{T}{T} \int_0^{T/2} F \, v \, dt = \Delta t \frac{T}{T} \int_{x(0)}^{x(T/2)} F \, dx \quad (68)
$$

$$
= -\frac{\Delta t}{T} (U(T/2) - U(0)) = 0 \quad (69)
$$
as $U$ has same value at each turning point

$\rightarrow$ energy conserved on average with Euler-Cromer for oscillatory motion
For comparison: with explicit Euler method $\Delta E_n$ contains term $\sum_{i=0}^{n-1} F_i^2$ which increases monotonically with $n$ and

$$\Delta E_n = -\frac{1}{8} \Delta t^2 (F_0^2 - F_n^2)$$  \hspace{1cm} (70)$$

with $v_0 = 0 \rightarrow F_0^2 \geq F_n^2 \rightarrow \Delta E_n$ oscillates between 0 and $-\frac{1}{8} \Delta t^2 F_0^2$ per cycle.

Energy is bounded as for Euler-Cromer, but $\langle \Delta E_n \rangle \neq 0$
Consider the following ODE

\[ \frac{dx}{dt} = -cx \]  \hspace{1cm} (71)

with \( c > 0 \) and \( x(t = 0) = x_0 \). Analytic solution is \( x(t) = x_0 \exp(-ct) \).

The explicit Euler method gives:

\[ x_{n+1} = x_n + \dot{x}_n \Delta t = x_n - cx_n \Delta t = x_n(1 - c\Delta t) \]  \hspace{1cm} (72)

So, every step will give \((1 - c\Delta t)\) and after \( n \) steps:

\[ x_n = (1 - c\Delta t)^n x_0 = (a)^n x_0 \]  \hspace{1cm} (73)

But, with \( a = 1 - c\Delta t \):

\[
\begin{align*}
0 < a < 1 & \quad \Rightarrow \quad \Delta t < 1/c \quad \text{monotonic decline of } x_n \\
-1 < a < 0 & \quad \Rightarrow \quad 1/c < \Delta t < 2/c \quad \text{oscillating decline of } x_n \\
a < -1 & \quad \Rightarrow \quad \Delta t > 2/c \quad \text{oscillating increase of } x_n
\end{align*}
\]  \hspace{1cm} (74)
Stability analysis of the Euler method II

Stability of the explicit Euler method for different $a = 1 - c\Delta t$

In contrast, consider implicit Euler method (Euler-Cromer):

$$x_{n+1} = x_n + \dot{x}_{n+1}\Delta t = x_n - cx_{n+1}\Delta t$$  \hspace{1cm} (75)

$$\Rightarrow x_{n+1} = \frac{x_n}{1 + c\Delta t}$$  \hspace{1cm} (76)

decreases for all $\Delta t$ (!)
The Euler-Richardson method

Sometimes it is better, to calculate the velocity for the midpoint of the interval:

**Euler-Richardson method ("Euler half step method")**

\[
\begin{align*}
    a_n &= \frac{F(x_n, v_n, t_n)}{m} \quad (77) \\
    v_M &= v_n + a_n \frac{1}{2} \Delta t \quad (78) \\
    x_M &= x_n + v_n \frac{1}{2} \Delta t \quad (79) \\
    a_M &= \frac{F \left( x_M, v_M, t_n + \frac{1}{2} \Delta t \right)}{m} \quad (80) \\
    v_{n+1} &= v_n + a_M \Delta t \quad (81) \\
    x_{n+1} &= x_n + v_M \Delta t \quad (82)
\end{align*}
\]

We need twice the number of steps of calculation, but may be more efficient, as we might choose a larger step size as for the Euler method.
Cromer, A. 1981, American Journal of Physics, 49, 455