Computational Astrophysics I: Introduction and basic concepts

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Applications:
The Lane-Emden equation
The Lane-Emden-Equation I

We remember:

**Example: Boundary values**

First two equations of stellar structure (e.g., for white dwarf)

\[
\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho} \quad \text{mass continuity} \quad (1)
\]

\[
\frac{\partial P}{\partial m} = -\frac{G M}{4\pi r^4} \quad \text{hydrostatic equilibrium} \quad (2)
\]

+ equation of state \( P(\rho) \) (e.g., ideal gas \( P(\rho, T) = RT\rho/\mu \)), and boundary values

- center \( m = 0 : r = 0 \) \quad (3)

- surface \( m = M : \rho = 0 \rightarrow P = 0 \) \quad (4)

→ solve for \( r(m) \), specifically for \( R* = r(m = M*) \)
Derivation

(see also Hansen & Kawaler 1994)

→ if equation of state (EOS) for pressure is only function of density, e.g., completely degenerate, nonrelativistic, electron gas

\[ P_e = 1.004 \times 10^{13} \left( \frac{\rho [g \text{ cm}^{-3}]}{\mu_e} \right)^{5/3} \text{ dyn cm}^{-2} \] \hspace{1cm} (5)

so, \( P \propto (\rho/\mu_e)^{5/3} \) power law ...

\( (\mu_e = [\sum Z_i X_i y_i/A_i]^{-1} \) mean molecular weight per electron, e.g.,

\[ \mu_e \approx (\frac{1.0.7.1}{1} + \frac{2.0.3.1}{4}) \approx 1.2 \text{ for fully ionized H-He plasma} \]

Polytropes are pseudo-stellar models where a power law for \( P(\rho) \) is assumed a priori without reference to heat transfer/thermal balance

→ only hydrostatic and mass continuity equation taken into account
define a polytrope as

\[ P(r) = K \rho^{1+1/n}(r) \]  

(6)

with some constant \( K \) and the polytropic index \( n \).

→ polytrope must be in hydrostatic equilibrium, so hydrostatic equation

\[
\frac{dP}{dr} = -\frac{GM_r}{r^2} \rho / \rho \cdot r^2 \frac{d}{dr}
\]

(7)

\[
\frac{d}{dr} \left( r^2 \frac{dP}{dr} \right) = -G \frac{dM_r}{dr} = -4\pi Gr^2 \rho
\]

(8)

with the continuity equation \( \frac{dM_r}{dr} = 4\pi r^2 \rho \) and the mass variable

\( M_r = \int_0^r dm(r) \), i.e., \( M_r = 0 \) → center \( (r = 0, \rho = \rho_c) \) and \( M_r = M_* \) → surface \( (r = R_*, \rho = 0) \)

so
\[ \frac{1}{r^2} \frac{d}{dr} \left( \frac{r^2}{\rho} \frac{dP}{dr} \right) = -4\pi G \rho \] (9)

→ Poisson’s equation of gravitation with \( g(r) = \frac{d\Phi}{dr} = \frac{GM_r}{r^2} \), and \( \frac{dP}{dr} = -\frac{GM_r}{r^2} \rho \rightarrow \nabla^2 \Phi = 4\pi G \rho \) in spherical coordinates

find transformations to make Eq. (9) \textit{dimensionless}. Define \textit{dimensionless} variable \( \theta \)

\[ \rho(r) = \rho_c \theta^n(r) \] (10)

→ then, power law for pressure from Eq. (6)

\[ P(r) = K \rho^{1+1/n}(r) = K \rho_c^{1+1/n} \theta^{n+1}(r) = P_c \theta^{1+n}(r) \] (11)

\[ \rightarrow P_c = K \rho_c^{1+1/n} \] (12)
inserting Eqs. (10) & (12) into Eq. (9)

\[
\frac{(n + 1)P_c}{4\pi G \rho_c^2} \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} \right) = -\theta^n
\]

(13)

together with dimensionless radial coordinate \(\xi\)

\[
r = r_n \xi \quad \text{with scale length} \quad r_n^2 = \frac{(n + 1)P_c}{4\pi G \rho_c^2}
\]

(14)

our Poisson’s equation (9) becomes

→ so called
The Lane-Emden-Equation VI

The Lane-Emden equation (Lane 1870; Emden 1907)

\[ \frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n \]  

(15)

with solutions “polytropes of index \(n\)” \(\theta_n(\xi)\)

applications:

- describe i.g. self-gravitating spheres (of plasma)
- Bonnor-Ebert sphere \((n \rightarrow \infty, \text{ so } u, e^{-u} \text{ instead of } \theta, \theta^n): \text{ stable, finite-sized, finite-mass isothermal cloud with } P \neq 0 \text{ at outer boundary} \rightarrow \text{Bonnor-Ebert mass (Ebert 1955; Bonnor 1956)}
- characterize (full) stellar structure models, e.g., Bestenlehner (2020)
- composite polytropic models for modeling of massive interstellar clouds with a hot ionized core, stellar systems with compact, massive object (BH) at centre
Remarks:

if EOS is ideal gas $P = \rho N_A k T / \mu$

$$P(r) = K' T^{n+1}(r), \quad T(r) = T_c \theta(r)$$

(16)

with $K' = \left( \frac{N_A k}{\mu} \right)^{n+1} K^{-n}$, $T_c = K \rho_c^{1/n} \left( \frac{N_A k}{\mu} \right)^{-1}$

(17)

→ polytrope with EOS of ideal gas and mean molecular weight $\mu$ gives temperature profile, radial scale factor is

$$r_n^2 = \left( \frac{N_A k}{\mu} \right)^2 \frac{(n + 1) T_c^2}{4 \pi G \rho_c} = \frac{(n + 1) K \rho_c^{1/n-1}}{4 \pi G}$$

(18)
Requirements for physical solutions:

central density $\rho_c \rightarrow \theta(\xi = 0) = 1$

spherical symmetry at center $(dP/dr|_{r=0}) \rightarrow \theta' \equiv d\theta/d\xi = 0$ at $\xi = 0$

→ suppresses divergent solutions of the 2nd order system → regular solutions (E-solutions)

surface $P = \rho = 0 \rightarrow \theta_n = 0$ (first occurrence of that!) at $\xi_1$

Boundary conditions for polytropic model

$\theta(0) = 1, \ \theta'(0) = 0$ at $\xi = 0$ (center)

$\theta(\xi_1) = 0$ at $\xi = \xi_1$ (surface)

So stellar radius

$$R = r_n \xi_1 = \sqrt{\frac{(n + 1)P_c}{4\pi G \rho_c^2}} \xi_1$$ (19)

for given $K, n$, and either $\rho_c$ or $P_c$ ($P_c = K \rho_c^{1+1/n}$)
Analytic E-solutions

→ analytic regular solutions exist for \( n = 0, 1, 5 \)

\( n = 0 \) constant density sphere, \( \rho(r) = \rho_c \), and

\[
\theta_0(\xi) = 1 - \frac{\xi^2}{6} \quad \rightarrow \quad \xi_1 = \sqrt{6}
\]  

(20)

so \( P(\xi) = P_c \theta(\xi) = P_c \left[1 - \left(\frac{\xi}{\xi_1}\right)^2\right] \). For \( P_c \) we need \( M, R \) from Eq. (19): \( P_c = \left(\frac{3}{8\pi}\right)\left(\frac{GM^2}{R^4}\right) \)

\( n = 1 \) solution \( \theta_1 \) is sinc function

\[
\theta_1 = \frac{\sin \xi}{\xi} \quad \text{with} \quad \xi_1 = \pi
\]

(21)

→ \( \rho = \rho_c \theta \) and \( P = P_c \theta^2 \)
$n = 5$ finite central density $\rho_c$ but infinite radius $\xi_1 \to \infty$:

$$\theta_5(\xi) = \frac{1}{\sqrt{1 + \frac{\xi^2}{3}}}$$

(22)

contains finite mass

solutions with $n > 5$ have also infinite radius, but also infinite mass
The Lane-Emden-Equation XI

For the interesting cases $0 \leq n \leq 5 \rightarrow$ numerical solution

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\theta}{d\xi} \right) = 2 \frac{d\theta}{\xi d\xi} + \frac{d}{d\xi} \frac{d\theta}{d\xi} = -\theta^n \quad (23)$$

Reduction: set $x = \xi$, $y = \theta$, $z = (d\theta/d\xi) = (dy/dx)$

$$y' = \frac{dy}{dx} = z, \quad (24)$$
$$z' = \frac{dz}{dx} = -y^n - \frac{2}{x}z \quad (25)$$

Assume that we have values $y_i$, $z_i$ at a point $x_i$, so that we can get with some step size $h$: $y_{i+1}$ & $z_{i+1}$ at $x_{i+1} = x_i + h$
Then with RK4:

\begin{align*}
k_1 &= h \cdot y'(x_i, y_i, z_i) = h \cdot (z_i) \\
\ell_1 &= h \cdot z'(x_i, y_i, z_i) = h \cdot (-y_i^n - \frac{2}{x_i}z_i) \\
k_2 &= h \cdot y'(x_i + h/2, y_i + k_1/2, z_i + \ell_1/2) = h \cdot (z_i + \ell_1/2) \\
\ell_2 &= h \cdot z'(x_i + h/2, y_i + k_1/2, z_i + \ell_1/2) \\
&= h \cdot \left(-\left(y_i + k_1/2\right)^n - \frac{2}{x_i + h/2}\left(z_i + \ell_1/2\right)\right) \\
k_3 &= h \cdot y'(x_i + h/2, y_i + k_2/2, z_i + \ell_2/2) \\
\ell_3 &= h \cdot z'(x_i + h/2, y_i + k_2/2, z_i + \ell_2/2) \\
k_4 &= h \cdot y'(x_i + h, y_i + k_3, z_i + \ell_3) \\
\ell_4 &= h \cdot z'(x_i + h, y_i + k_3, z_i + \ell_3) \tag{35}
\end{align*}
Integration could be started for $\xi = 0$, as

$$\theta_n(\xi) = 1 - \frac{\xi^2}{6} + \frac{n}{120} \xi^4 - \frac{n(8n - 5)}{15120} \xi^6 + \ldots$$

(36)

$$\rightarrow \theta'_n(\xi) = -\frac{1}{3} + \frac{n}{40} \xi^3 - \frac{n(8n - 5)}{2520} \xi^5 + \ldots$$

(37)

So for $\xi \rightarrow 0$ then $y' \rightarrow -1/3$

but there is a irregularity in $\xi = 0$ in $z' = -y^n - \frac{2}{\chi}z$ (Eq. (25))

better: choose $0 < \xi \ll 1$ and compute with help of Eq. (36) $y, y'(= z), z'$
construct polytropes for $n < 5$ and given $M$, $R$
→ possible as long as $K$ not fixed
because of definition of $\theta$ from $\rho(r) = \rho_c \theta^n(r)$ (Eq. (10)) and $r = r_n \xi$ (Eq. 14)

$$m(r) = \int_0^r 4\pi \rho r^2 dr = 4\pi \rho_c \int_0^r \theta^n r^2 dr = 4\pi \rho_c \frac{r^3}{\xi^3} \int_0^\xi \theta^n \xi^2 d\xi$$  (38)

note that $r^3/\xi^3 = r_n^3$ is constant. From Lane-Emden equation (15)

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n \rightarrow \theta^n \xi^2 = -\frac{d}{d\xi} \left( \xi^2 \frac{d\theta}{d\xi} \right)$$

follows direct integration, so

$$m(r) = 4\pi \rho_c r^3 \left( -\frac{1}{\xi} \frac{d\theta}{d\xi} \right)$$  (39)

→ Eq. (39) contains $\xi$ and $r$, related by Eq. (14): $r/\xi = r_n = R/\xi_1$, so for the surface:
Applying the Lane-Emden equation to stars II

\[ M = 4\pi \rho_c R^3 \left( -\frac{1}{\xi} \frac{d\theta}{d\xi} \right)_{\xi=\xi_1} \quad (40) \]

With help of the mean density \( \bar{\rho} := M/(\frac{4}{3}\pi R^3) \) this can be written as

\[ \frac{\bar{\rho}}{\rho_c} = \left( -\frac{3}{\xi} \frac{d\theta}{d\xi} \right)_{\xi=\xi_1} \quad (41) \]

Note the right hand side depends only on \( n \), can be computed. E.g., for \( n = 1 \rightarrow ( -\frac{3}{\xi} \frac{d\theta}{d\xi} )_{\xi=\xi_1} = 1 \)

the larger \( n \rightarrow \) the smaller \( \frac{\bar{\rho}}{\rho_c} \rightarrow \) the higher the density concentration
Ebert, R. 1955, ZAp, 37, 217
Emden, R. 1907, Gaskugeln
Lane, H. J. 1870, American Journal of Science, 50, 57