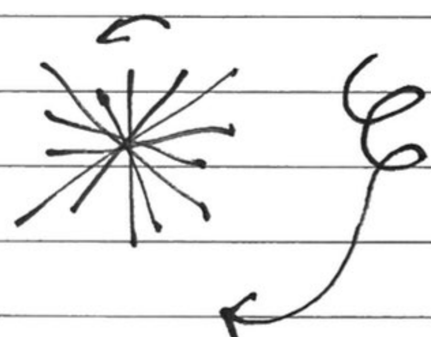
Satz von Liouville

Phasenraumvolumen mech. Systeme zeitlich konstant.

(1)

Mech. Sys.: System mit n-Freiheitsgraden macht Trajektorie im Phasenraum (dim 2n)

2n Achsen



(Gibbs)

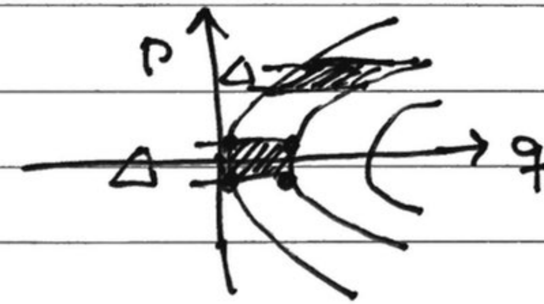
$t=0$: $q = 1, p = 0$

$q = 0.9, p = 0$

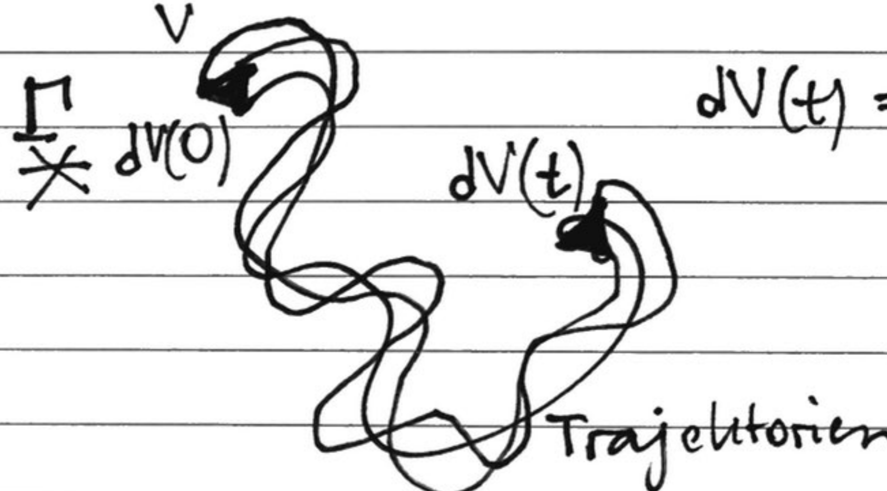
$q = 0.9, p = 0{,}1$

$q = 1{,}0, p = 0{,}1$

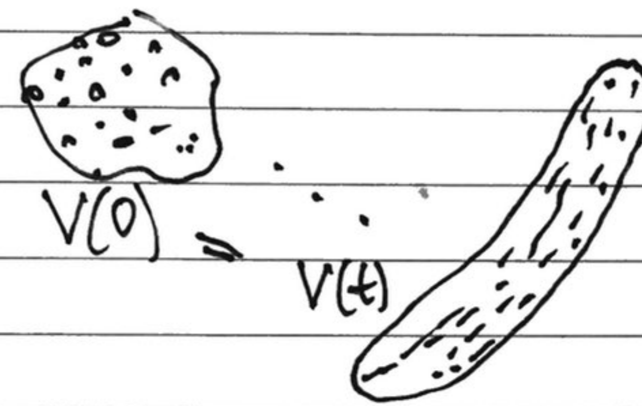
$A(0) = A(T/2)$



beide schraffierten Flächen gleich



$dV(t) = \text{const}$



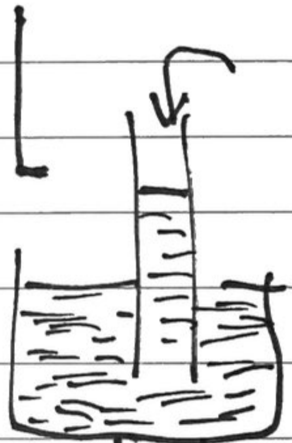
$\mathcal{V}$

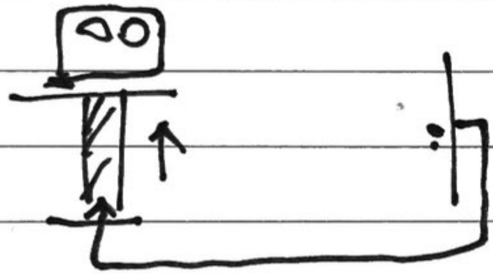
$V = \text{mes}(\mathcal{V})$

② inkompressible Flüssigkeit

$V = \text{const}$

Autowerkstatt





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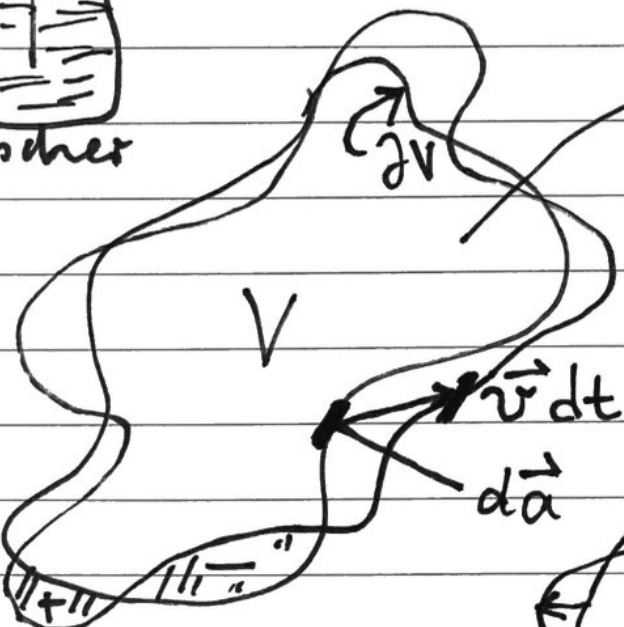
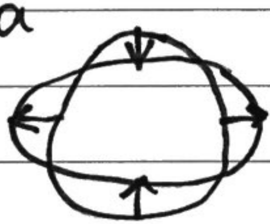
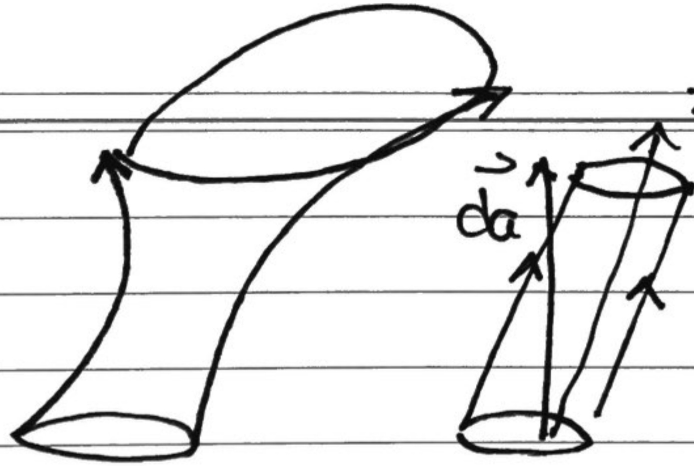






$$dV = d\vec{a} \cdot d\vec{s}$$

$$0 = \delta V = \oint_{\partial V} \underbrace{d\vec{a} \cdot \vec{v}\,\delta t}_{dV}$$

$$= \delta t \oint_{\partial V} d\vec{a} \cdot \vec{v}$$

Satz v. Gauß

$$= \delta t \int_V dV \; \operatorname{div} \vec{v} \qquad (\nabla \cdot \vec{v})$$

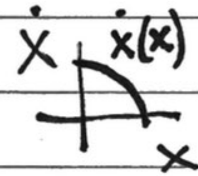
$$\Rightarrow \boxed{\nabla \cdot \vec{v} = 0}$$

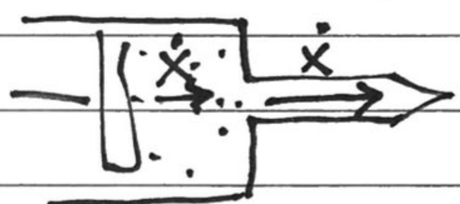
③ Idee von Liouville

$$0 = \nabla\cdot\vec{v}\,, \quad \vec{v} = \dot{\vec{r}} = \begin{pmatrix}\dot x\\ \dot y\\ \dot z\end{pmatrix}$$

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$0 = \frac{\partial \dot x}{\partial x} + \frac{\partial \dot y}{\partial y} + \frac{\partial \dot z}{\partial z}$$





$$\frac{\partial \dot x}{\partial x} \neq 0$$

jetzt im Phasenraum

$q_1, \dots, q_n, p_1, \dots, p_n$ ersetzt x, y, z

$\dot q_1, \dots, \dot q_n, \dot p_1, \dots, \dot p_n$ ersetzt $\dot x, \dot y, \dot z$

$$\sum_{i=1}^{n}\left(\frac{\partial \dot q_i}{\partial q_i} + \frac{\partial \dot p_i}{\partial p_i}\right) = \nabla\cdot\vec{\Phi}$$

$$= \sum_{i=1}^{n}\left(\frac{\partial}{\partial q_i}\left(\frac{\partial H}{\partial p_i}\right) + \frac{\partial}{\partial p_i}\left(-\frac{\partial H}{\partial q_i}\right)\right)$$

$$= \sum_{i=1}^{n}\left(\frac{\partial^2 H}{\partial q_i\,\partial p_i} - \frac{\partial^2 H}{\partial p_i\,\partial q_i}\right)$$

$$= 0$$

$$\nabla\cdot\vec{\Phi} = 0$$

$\vec{\Phi}$ = Phasenraumgeschwindigkeit

$$\vec{\Phi} = (\dot q_1, \dots, \dot q_n, \dot p_1, \dots, \dot p_n)$$

denn Divergenz im Phasenraum ist

$$\nabla = \left(\frac{\partial}{\partial q_{1x}}, \dots, \frac{\partial}{\partial q_{nz}}, \frac{\partial}{\partial p_{1x}}, \dots, \frac{\partial}{\partial p_{nz}}\right)$$

zu umständlich

$$\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_{n+1}}, \dots \frac{\partial}{\partial x_{2n}}\right)$$

$$\begin{pmatrix} q_i \\ p_i \end{pmatrix} \rightsquigarrow (x_j)$$

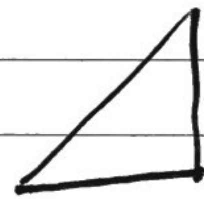
dann ist

$$\sum_{i=1}^{n} \left(\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i}\right) = \sum_{j=1}^{2n} \frac{\partial \dot{x}_j}{\partial x_j}$$

4 Durchrechnung:

div im $\mathbb{R}^3$ hart. war Vorkurs
jetzt im Phasenraum

auch in Greiner ...
auch in Khinchin
auch in Huang, Kerson
...



Simplex

Trick: Phasenraum hat Paare

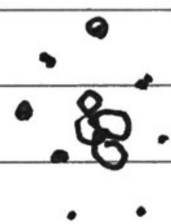
$q_1, \dots q_i, \dots q_n\; p_1 \dots p_i, \dots, p_n$

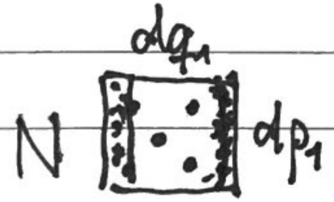
also Beweis einfacher als $\nabla\cdot$ in $\mathbb{R}^3$, da nur noch $\underbrace{\mathbb{R}^2 \times \dots \times \mathbb{R}^2}_{n}$

$$dV = dp_1\, dq_1 \dots dp_n\, dq_n$$

$$\left.\frac{\delta(dN)}{\delta t}\right|_1 =$$

$$n = dN/dV$$





N verschiedene Anfangsbed.

$$\left.\frac{\delta(dN)}{\delta t}\right|_i = -n\left(q_1, \dots, q_i + \frac{dq_i}{2}, \dots, q_n, p_1, \dots, p_n\right) \underbrace{dq_1 \dots \widehat{dq_i} \dots dq_n\, dp_1 \dots dp_n}_{= dq_1 \dots \cancel{dq_i} \dots dq_n\, dp_1 \dots dp_n}$$

$$\frac{\delta q_i}{\delta t}\big(\text{ alles nochmal }\big)$$